# Non-Trivial Pursuit: a Game of Cops and Robbers

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## 1 Introduction

Cops and Robbers is a two-player pursuit game played on an undirected, reflexive graph $G$. Player 1 controls $k$ cops who begin on (not necessarily distinct) vertices of $G$. Player 2 then places a single robber on another vertex. Each player takes turns moving their cops or their robber along edges to adjacent vertices, with the cops moving first.

The goal of the cops is to occupy the same vertex as the robber (capture him), while the robber's goal is to evade capture indefinitely.

We prove upper and lower bounds on the cop number of graphs with retract-decomposable structure. We then present an upper bound on the smallest $k$-cop-win graph for a given $k$, namely $5 \cdot 2^k$ (the Meyniel-extremal bound provided by finite projective planes only extends to prime powers, or so is conjectured) through a method called Petersenication, based on the Petersen graph. We study the graphs of the truncated Platonic solids, determining their cop numbers exactly. Finally, we improve an existing computer algorithm for determining cop number and discuss some experimental results, including the cop number of expander graphs.

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# 2 Retract-Decomposable Graphs

## 2.1 Retracts

Retractions are helpful in analyzing the cop number of graphs. If a graph $G$ has a subgraph $H$ that is a retract of $G$, we can say several things. Berarducci and Intrigila [4] showed that the cop number of $G$ is at least the cop number of $H$, since the robber can just stay in $H$ and there are no shortcuts for the cops to skip across areas of $H$ that they couldn't reach traveling in $H$.

If the robber is somewhere in $G$, we can define the robber's *shadow* on $H$ as the position in $H$ that the robber's vertex $G$ is mapped to by a retraction mapping. If a cop is on the robber's shadow, then the cop can always match the robber's move on $H$ and stay with the robber's shadow. The retraction mapping is the identity mapping when restricted to $H$, so if the robber enters $H$, it is its own shadow, and is captured by the cop. So, an appropriately placed cop can prevent the robber from entering $H$ without losing.

Geodesic paths are particularly useful examples of this, as shown by Aigner and Fromme [3]. Any geodesic path is a retraction of the graph it is in, where all vertices distance $n$ from one endpoint of the path are mapped to the vertex on path that is distance $n$ from that endpoint. If the distance is longer than the length of the path, the mapping is to the other endpoint. Since a path is copwin, a single cop can capture the robber's shadow on the path and then follow it, keeping the robber out of that path regardless of initial position and without help from other cops.

## 2.2 Retract Decomposition

Given a graph $G$, suppose $R$ is a family of subsets of the vertices of $G$ and $X$ is a connected graph with elements of $R$ as nodes. $(R, X)$ is a *retract decomposition* of $G$ if and only if:

- Every vertex of $G$ is in some element of $R$
- Every edge of $G$ has both its endpoints in some single set of $R$
- For any vertex $a \in G$, The subgraph of $X$ induced on the set of elements of $R$ that contain $a$ is connected

- For each element $A \in R$, The subgraph of $G$ induced on the elements of $A$, refered to as $G[A]$, is a connected retract of $G$

A strategy for a cops and robbers game on a graph can be determined from the strategy for an easier to analyze game on its decomposition. A strategy for the robber can be determined from a robber strategy for a single node of the decomposition. This allows retract decomposition to give both an upper and lower bound on cop number. The easier to analyze game is described below, and the bounds and their proof is given afterward.

## 2.3 Helicopter Cops and Robbers

Helicopter cops and robbers is a game played on a graph. The graph has a robber region and several nodes guarded by cops. The cop player's move consists of adding a cop to any node of the graph or removing a cop from anywhere on the graph. The robber region must be connected and cannot contain any cop nodes. Whenever the cops make a placement that disconnects the robber region, the robber must choose which component of the current robber region to make the new robber region. The robber loses if a cop placement covers the entire robber region.

A helicopter cops and robbers *cop strategy* on a graph $G$ is a series of cop placements and removals for every possible set of robber choices. A *k-cop winning strategy* is a cop strategy that guarantees a cop victory, while never having more than $k$ cops on the graph.

[Seymour and Thomas] show that that for a graph $G$ with tree width $k$, $k+1$ cops have a winning strategy for helicopter cops and robbers on $G$.

## 2.4 Cop Number Bounds

**Theorem 1.** *Given a graph $G$ and a retract decomposition of it $(R, X)$*

$$\max_{A \in R}\{C(G[A])\} \leq C(G) \leq \max_{A \in R}\{C(G[A])\} + tw(X) \tag{1}$$

*where $tw(X)$ is the tree-width of $X$.*

*Proof.* For every node $A \in R$, $G[A]$ is a retract of $G$, so $C(G[A]) \leq C(G)$. This proves the first inequality.

To avoid ambiguity, points of $G$ will be refered to only as *vertices* while points of $X$ will be refered to as *nodes*.

The helicopter cops' winning strategy for $tw(X) + 1$ cops can be used to formulate a cop strategy on $G$ for $\max_{A \in R}\{C(G[A])\} + tw(X)$ cops. The subgraphs induced by $G$ on each node $N$ of $X$ are retracts of $G$, so $C(G[N])$ cops can beat the robber's shadow on $G[N]$. The cop who captured the shadow can stay in $N$, moving with the shadow and preventing the robber from entering the vertices contained in the node. It acts as a guard in a helicopter cops and robbers game on $X$.

Separate the cops into two groups: $tw(X) + 1$ *guards* and $\max_{A \in R}\{C(G[A])\} - 1$ *helpers*. The cops can be initially placed anywhere.

When the helicopter strategy says to place a cop at node $A$ in $X$, move one of the guard cops and all of the helper cops to the subgraph $G[A]$ of $G$. Since there are $\max_{A \in R}\{C(G[A])\}$ cops on $G[A]$, which is at least the cop number of $G[A]$, the cops can catch the robber's shadow on $G[A]$. Once they do, the cop which is on the robber's shadow is considered to be *guarding* node $A$. If that cop was not in the guard group it is moved to the guard group while the original guard is moved to the helper group, so the size of each group is constant. Until the strategy calls for removing the cop from node $A$, the cop guarding $A$ moves with the robber's shadow on $A$, preventing the robber from moving through $A$.

When the helicopter strategy says to remove a cop from a node, stop guarding that node and allow the cop that was guarding it to be used to place on a new node.

Define $X'$ as $X$ minus all nodes being guarded and $G'$ as $G$ minus all vertices contained in guarded nodes. The helicopter cops and robbers strategy is played as if the robber region is the component of $X'$ that contains a node that contains the robber vertex.

For this strategy to work, the robber region defined must act like a robber region in a helicopter cops and robbers game. That is, a robber cannot change the robber region by moving somewhere that would correspond to a different component of $X'$.

We can prove this by proving that as long as the robber is not in a guarded node, if there is a path $P_G$ in $G'$ between the robber vertex $a$ and some $b \in G'$, there exists a path $P_X$ in $X'$ between every two nodes $A, B \in X'$ such that $a \in A$ and $b \in B$, which we can prove by induction on the path length.

For the base case, suppose that we have a length zero path from the robber vertex to itself. By the definition of retract decomposition, the set of nodes that contain the robber vertex is connected, so there is a path $P_X$ in $X'$ from any node that contains the robber vertex to any other containing the robber vertex. Since the robber vertex is not in a guarded node, none of the nodes in $P_X$ are guarded so all are in $X'$.

For the induction case, suppose that if there is a length $n$ path in $G'$ from the robber vertex $a$ to any vertex $b$, there is a path in $X'$ from any node $A$ that contains $a$ to any other node $B$ that contains $b$.

Now consider a vertex $c$ that is distance $n+1$ from $a$ in $G'$. There is a path $P_{n+1}$ from $a$ to $c$ in $G'$, and a path $P_n$ from $a$ to $b$ where $b$ is the vertex in $P_{n+1}$ that is distance $n$ from $a$. There exists some node $C$ that contains both $b$ and $c$. By the induction hypothesis, there is some path $P_{X1}$ in $X'$ that goes from any node that contains $a$ to the node $C$. From the base case proof, there is some path $P_{X2}$ in $X'$ that goes from any node that contains $c$ to the node $C$, since there is a length zero path from $c$ to $c$. Joining $P_{X1}$ and $P_{X2}$ gives a path from any node containing $a$ to any node containing $c$.

By playing the helicopter cops and robbers strategy, the cops will eventually restrict the robber region to a single node and place a guard on that node. When the cops beat the robber's shadow on that node, they will have actually captured the robber, since the robber can't leave the robber region.

This gives a winning strategy for $\max_{A \in R}\{C(G[A])\} + tw(X)$ cops on $G$, which proves the second inequality.

□

# 3 Petersenication

Our goal is to create a family of graphs with unbounded cop number using a recursive construction process. The order of these graphs will be an upper bound on the order of the smallest graph on which $k$ cops are necessary to win. It is a smaller upper bound than the one presented by Aigner and Fromme [3].

## 3.1 Construction

This process consists of joining together 5-cycles (pentagons) by 2 distinct Petersenic permutation mappings. A Petersenic mapping is one that maps vertices of pentagon $P_i$ to vertices of pentagon $P_j$ in such a way that the added edges form a Petersen Graph on the vertices $V(P_i) \cup V(P_j)$. Of course, from the properties of Petersen graphs, we know that there are no 3- or 4-cycles in such a graph. If we label the vertices of each pentagon $0, 1, 2, 3, 4$ consecutively, then two such mappings are $\Gamma = (1342)$ and $\Delta = (0143)$.

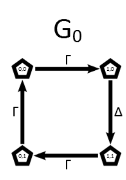
So if we map the vertices from $P_i$ to the vertices of $P_j$ by $\Gamma$, this is represented by a set of five edges. These 5 edges connect vertex 0 of $P_i$ to vertex 0 of $P_j$, vertex 1 of $P_i$ to vertex 3 of $P_j$, vertex 2 of $P_i$ to vertex 1 of $P_j$, Vertex 3 of $P_i$ to vertex 4 of $P_j$, and vertex 4 of $P_i$ to vertex 2 of $P_j$.

Also, note that if pentagon $P_i$ is mapped to pentagon $P_j$ by $\Gamma$ or $\Delta$, then pentagon $P_j$ is mapped to pentagon $P_i$ by $\Gamma^{-1} = (1243)$ or $\Delta^{-1} = (0341)$, respectively. Note that each mapping (and inverse) has cycle structure of one length 4-cycle. This implies not only that $\Gamma^3 = \Gamma^{-1}$ and $\Delta^3 = \Delta^{-1}$, but also that exactly one vertex in pentagon $P_i$ ($P_j$) would be mapped to a vertex in pentagon $P_j$ ($P_i$) of the same index.

We must be careful in choosing 2 mappings $\alpha$ and $\beta$ of the possible Petersenic mappings. They must satisfy the condition that $\alpha\beta^{-1}(i) \neq i$ for every $i \in [0, 4]$. If true, its cycle structure is either one 5-cycle or one 2-cycle and one 3-cycle; this structure, and thus the condition, is shared by its inverse: $\beta\alpha^{-1}$. Also since cycle structure is preserved under conjugation, $\alpha^{-1}(\alpha\beta^{-1})\alpha = \beta^{-1}\alpha$ and its inverse $\alpha^{-1}\beta$ share the same condition. Thus if the condition is satisfied by one of $\alpha\beta^{-1}$, $\beta\alpha^{-1}$, $\beta^{-1}\alpha$, $\alpha^{-1}\beta$, it is satisfied by all of them.

The $\Gamma$ and $\Delta$ from above satisfy this condition: $\Gamma\Delta^{-1} = \Gamma(0341) = (04321)$, and we will use these mappings throughout the rest of the construction and proof.

**Start of the Process:** Let $G_0$ consist of 4 pentagons. Instead of the normal graph portrayal, we will represent $G_0$ visually by replacing the 5 edges between any two connected pentagons with a directed arrow labeled either by $\Gamma$ or $\Delta$ and condensing each pentagon (5 vertices and 5 edges in the normal graph representation) into a single vertex. These arrows indicate the direction of the mapping. Opposing this direction indicates the mapping's inverse.



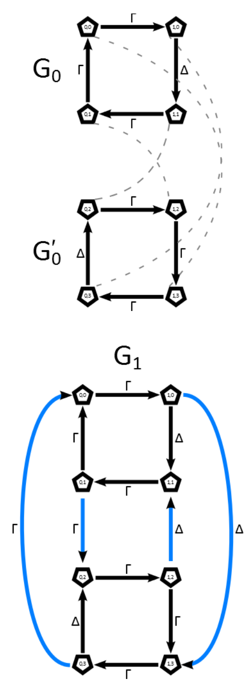

Let $G_0$ be 4 pentagons joined by directed mappings $\Gamma$ or $\Delta$ with a labeling $L$ of the vertices. The labeling $L$ assigns ordered pairs from $\{0, 1\}$ to each vertex of $G_0$ in such a way that pairs which differ in exactly one coordinate are adjacent. To each vertex $v$ of $G_0$, we associate a pentagon $P_v$. We write $P_{x,y}$ for the pentagon associated to the vertex having label $(x, y)$.

Assume that $G_i$ has been defined. Let $G'_i$ be a copy of $G_i$ so that the vertex $v'$ of $G'_i$ denotes the vertex which corresponds to the vertex $v$ of $G_i$. Let $L'$ be the labeling of the vertices of $V(G'_i)$ given by

$$L'(v') = (1 - x, 2^{i+2} - 1 - y) \iff L(v) = (x, y).$$

.

**Recursion:** For $P_{x,y}$ in the original graph $G_i$, map every vertex of that pentagon to a unique vertex of pentagon $P_{L'(P'_{(1-x,y)})}$. If $x = 0$, then let this mapping be $\Gamma^{(-1)^{y+1}}$ and if $x = 1$, then let this mapping be $\Delta^{(-1)^y}$. Call the resulting graph $G_{i+1}$ and use the vertex labelings induced by $L$ and $L'$.




**Lemma 1:** In a Petersenic construction, any pair of pentagons that are connected and share the same x-coordinate must have y-coordinates of opposite parity.

*Base Case:* For the graph $G_0$, we have that $P_{0,0}$ connects to $P_{0,1}$ and $P_{1,0}$ connects to $P_{1,1}$.

*Induction:* Given that the lemma holds for $G_k$, we see that the only mappings we have to check are those to $G'_k$ that we add for $G_{k+1}$. These are of the form $P_{x,y} \in G_k \to P_{x,2^{k+2}-1-y}$. Because $y$ and $2^{k+2}-1-y$ are necessarily of different parity, we see that our claim holds for $G_{k+1}$. □

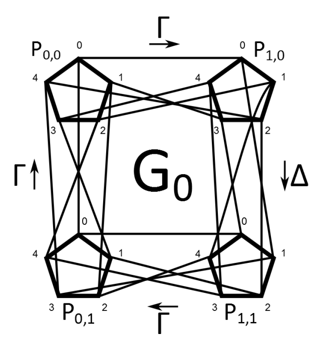
**Lemma 2:** In a Petersenic construction, if there exists a mapping $S_i \in S = [\Gamma, \Gamma^{-1}, \Delta, \Delta^{-1}]$ from $P_{x,y}$ to $P_{x,z}$, then there exists a mapping $S_{i+2(mod 4)}$ from $P_{1-x,z}$ to $P_{1-x,y}$.

*Base Case:* For the graph $G_0$, we see that the mapping from $P_{1,0}$ to $P_{1,1}$ is $\Delta = S_2$. We also see that the mapping from $P_{0,1}$ to $P_{0,0}$ is $\Gamma = S_0$.

*Induction:* Given that the lemma holds for $G_k$, we only need to consider the mappings that we add to create $G_{k+1}$. We map a pentagon $P_{1,y}$ with $\Delta^{(-1)^y}$ to pentagon $P_{1,2^{k+2}-1-y}$. Also, since we map a pentagon $P_{0,y}$ with $\Gamma^{(-1)^{y+1}}$ to pentagon $P_{0,2^{k+2}-1-y}$, then the mapping we are concerned with, from $P_{0,2^{k+2}-1-y}$ to $P_{0,y}$ is $\Gamma^{(-1)^{(y+1)+1}} = \Gamma^{(-1)^y}$ . Therefore, these pairs of mappings are either $\Delta$ and $\Gamma$ or $\Delta^{-1}$ and $\Gamma^{-1}$, respectively. □

**Theorem 2.** *For all $k \geq 0$, there exists a graph $G_k$ of order $5 \cdot 2^{k+2}$ such that $G_k$ is $(k+4)$ - regular, and has no 3- or 4-cycles. $G_k$ satisfies the above Petersenic construction.*

*Base Case:* Picture Proof.




*Induction:* Assume that $(k+4)$-regular graph $G_k$ of order $5 \cdot 2^{k+2}$ was created according to the above construction and contains no 3- or 4-cycles. Then, construct $G_{k+1}$ by the above construction, namely creating a copy $G'_k$, rotating this copy, and then adding in the appropriate mappings.

First, we see that $G_{k+1}$ contains only the vertices of $G_k$ and $G'_k$, giving it order $2 \cdot 5 \cdot 2^{k+2} = 5 \cdot 2^{(k+1)+2}$.

We also note that pentagon $P_{x,y} \in G_k$ is mapped to $P_{x,2^{k+1}-1-y} \in G'_k$. Because $f(y) = 2^{k+1} - 1 - y$ is a bijective function (in fact it is its own inverse), we see that every pentagon in $G_k$ maps to exactly one pentagon of $G'_k$ and every pentagon in $G'_k$ maps to exactly one pentagon of $G_k$. Therefore, the degree of every vertex in the graph is increased by one, and the graph is $((k+1)+4)$-regular.

A 4-cycle in $G_{k+1}$ cannot contain vertices from only 2 of the pentagons. This is because if there is a mapping between any two pentagons, then the induced subgraph on the union of their vertices must be a Petersen graph, which we know has no 4-cycles. Likewise, we know that a 4-cycle cannot contain vertices from only 3 of the pentagons. We first note that if there are 3 pentagons which are all joined to one another, then they must share the same x-coordinate. But by Lemma 1, all pentagons with the same x-coordinate can only be mapped to pentagons with opposite y-coordinate parity. But, there is no way to choose three y-coordinates that all have different parity, so we see that this case cannot exist. Note that the above argument also shows that there can be no 3-cycle in this construction, as the arguments for why a 4-cycle cannot occur on a pair of pentagons or on exactly three of them holds for 3-cycles.

So, we must only show that a 4-cycle cannot occur on 4 different pentagons on $G_{k+1}$. We see that there are 2 types of such 4-cycles that can be formed on this graph.

The first case occurs when we have mappings from pentagons of the form $(P_{x,y_1}, P_{x,2^{k+2}-1-y_1}, P_{x,2^{k+1}-1+y_2}, P_{x,y_2})$. We know from the rotation of $G'_k$ in the construction that $P_{x,y_1}$ is mapped to $P'_{1-x,y_1} = P_{x,2^{k+2}-1-y_1}$ in $G_{k+1}$. It then follows that the mapping from $P_{1-x,y_1}$ to $P_{1-x,y_2}$ is the same as the one from $P_{x,2^{k+2}-1-y_1}$ to $P_{x,2^{k+2}-1-y_2}$. With this knowledge, we can conclude from Lemma 2 that the mapping from $P_{x,y_2}$ to $P_{x,y_1}$ must be the opposite mapping of the one between $P_{x,2^{k+2}-1-y_1}$ and $P_{x,2^{k+2}-1+y_2}$. By Lemma 1 and the construction, we know that because $y_1$ and $y_2$ have opposite parity, the new mappings originating from them in $G_{k+1}$ are inverses of each other. Because we need one of these mappings to go from $G_k$ to $G'_k$ and the other from $G'_k$ to $G_k$, we consider these two mappings to be in the same direction. So, the 4-cycle mapping on these pentagons is $\phi\alpha\phi\beta$ for $\phi, \alpha, \beta \in \{\Delta, \Gamma, \Gamma^{-1}, \Delta^{-1}\}, \alpha \neq \beta$. Without loss of generality, we see that either $\alpha = \phi$ or $\alpha = \phi^{-1}$. We see in both cases that $\phi\alpha\phi\beta = \alpha^{-1}\beta$. We know from the construction that all permutations of the form $\alpha^{-1}\beta$ must be 5-cyclic permutations.

The second case occurs the cycles that have pentagons of the form ($P_{x,y}$, $P_{1-x,y}$, $P_{1-x,2^{k+2}-1-y}$, $P_{x,2^{k+2}-1-y}$). From the construction, we see that mapping between the first two pentagons and the mapping between the final two should both be $\Gamma$ or $\Gamma^{-1}$. From Lemma 2, the mapping from $P_{1-x,y}$ to $P_{1-x,2^{k+2}-1-y}$ should be the opposite of the one from $P_{x,2^{k+2}-1-y}$ to $P_{x,y}$. Thus, we have a mapping of the form $\gamma\alpha\gamma\beta$, which we know to be a specific case of the mapping reduced in the previous case. Thus, we see that this construction can contain no 4-cycles.

**Corollary:** For $c \geq 3$, there exists a $c$-regular graph $G$ with $|G| = 5 \cdot 2^{c-2}$ such that $c(G) \geq c$.

*Proof:* Follows from Aigner and Fromme [3] Theorem.

# 4 Polyhedra

The regular convex polyhedra (or Platonic solids) are central objects of interest in mathematics. It is an interesting exercise to compute their cop numbers, as well as the cop numbers of the truncated regular polyhedra.

The tetrahedron is $K_4$ and thus is copwin. The cube, octahedron, and icosahedron are all covered by 2 cops and are not dismantlable, and so are 2-cop win. The dodecahedron is planar, 3-regular and girth 5, so by Aigner and Fromme [3] it is 3-cop win. This exact strategy also applies to the truncated icosahedron.

We proved the cop number for the five truncated regular polyhedra: the icosahedron, as above, is 3, and the cube, tetrahedron and octahedron 2. The truncated dodecahedron, proved below, is 3. We used geodesic paths to cover the cubeand tetrahedron in a way that guarantees the cops a winning strategy, using a refined version of the popular greedy method. Geodesic paths are 1-guardable, as shown by Aigner and Fromme [3] . If 2 cops start on opposite vertices on the truncated octahedron, a case-by-case strategy shows that they can win:

### 4.0.1 Truncated Octahedron

*Proof.* Label vertices as shown. Start with a cop on 1 and a cop on 16. Clearly the robber cannot start on 1, 2, 4, 7, 13, 15, 16, or 22 (this accounts

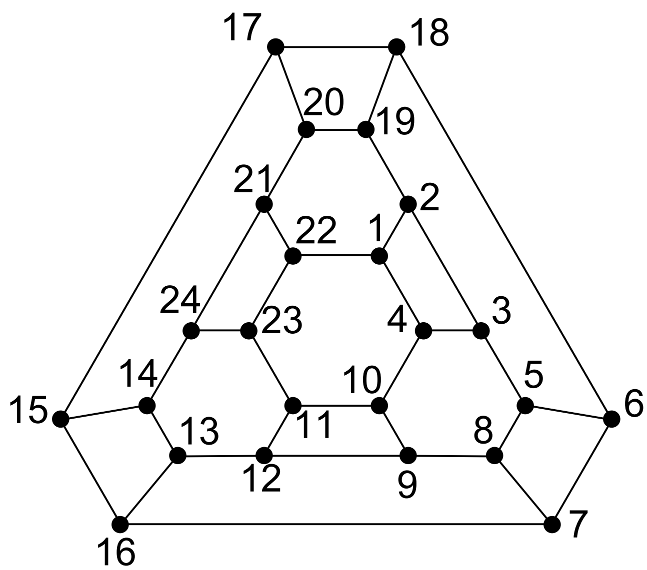



Figure 1: The truncated octahedron

for 8/24 vertices). If the robber starts on 14, move Cop 1 (the cop at 1) to 22. If the robber stays put he is trapped. If he moves to 24, Cop 2 (the cop at 16) moves to 13 and traps him. Thus 14 is a bad starting point, and by symmetry, so is 3 (10/24 vertices). If the robber starts on 24, Cop 1 moves to 22 and Cop 2 moves to 13: trapped again. This makes 24 and 5 bad starting points (12/24 vertices).

If the robber starts at 23, Cop 1 moves to 22, Cop 2 moves to 13. The Robber then must move to 11, as moving to 24 results in entrapment. Cop 2 then moves to 12 while Cop 1 stays put. The Robber must move to 10. Then Cop 1 moves back to 1 and the Robber is trapped. This strategy applies to vertex 23, 21, 6, and 8 (16/24 vertices).

Suppose the Robber starts at 10. Then if Cop 1 moves to 2 and Cop 2 moves to 13, then no matter where the robber goes (10, 4, 9, 11) the new position is equivalent to one of the ones previously discussed, and the robber will be captured. This strategy applies to 10, 12, 18, and 20 (20/24 vertices).

Now if the robber starts at 11, Cop 1 moves to 22 and Cop 2 moves to 13. If the robber stays at 11, Cop 2 moves to 12 and forces the robber to move to 10. Then Cop 1 moves to 1 and the robber is trapped. If the Robber instead does not stay at 11 but moves to 10, Cop 1 moves to 1 and Cop 2 moves to 12. Again the robber is trapped. This strategy works for all of the remaining vertices.

Hence the truncated octahedron is 2-cop-win. □

### 4.0.2 Truncated Dodecahedron

Let G be the truncated dodecahedron. Prove that $c(G) = 3$:

*Proof.* Since G is planar, we know by Aigner and Fromme [3] that that $c(G) \leq 3$. Now we show that 2 cops do not suffice.

Let $d(C_i, R)$ be the distance from a given cop to the robber along a geodesic path, where $C_1$ is the cop with the shortest path, and $C_2$ is the cop with the longest path. Before any given round in the game, the robber occupies one vertex of some 3-cycle(triangle) in the graph. Call this vertex R and this triangle the Robber's Triangle. Each of the cops' geodesics to R must pass through (or terminate at) the Robber's Triangle. Let $T_{C_1}$ be the vertex of $C_1$'s geodesic to R that is distance 1 from R, and likewise let $T_{C_2}$ be $C_2$'s vertex (see Figure 2). Also, let the 'truncated girth' of the graph be the girth of the graph not including the 3-cycles formed by truncation. For the truncated dodecahedron, the truncated girth is 10.

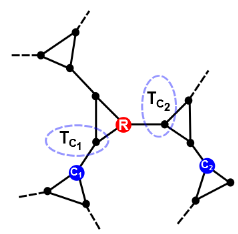
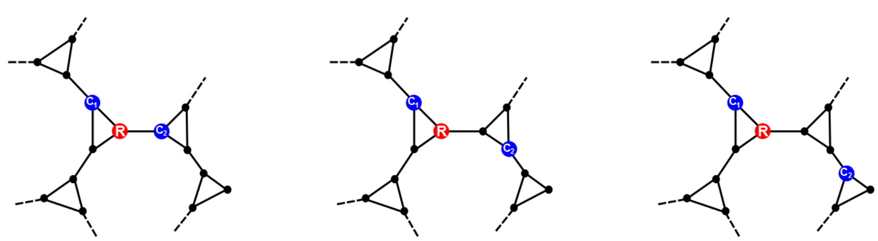



Figure 2: Example of $T_{C_1}$ and $T_{C_2}$

Define a "bad" robber position as one in which BOTH of the following are true (since the robber is trapped and can always be caught from this position - see Figure 3):

- $d(C_2, R) \leq 3$ where $T_{C_2}$ is not on Robber's Triangle
- $d(C_1, R) = 1$ where $T_{C_1}$ is on Robber's Triangle



(a) $d(C_2, R) = 1$ (b) $d(C_2, R) = 2$ (c) $d(C_2, R) = 3$

Figure 3: "Bad" Positions

Let a "good" robber position be any position which is not "bad".

We now show that from a "good" starting position, one round can be played and the robber will be in a new "good" position. That is, the robber can take one specific move and the cops can take any one move so that the robber's position is still "good".

**Case 1:** $d(C_1, R) \geq 2$ and $d(C_2, R) \geq 5 \quad \rightarrow \quad$ Robber does not move

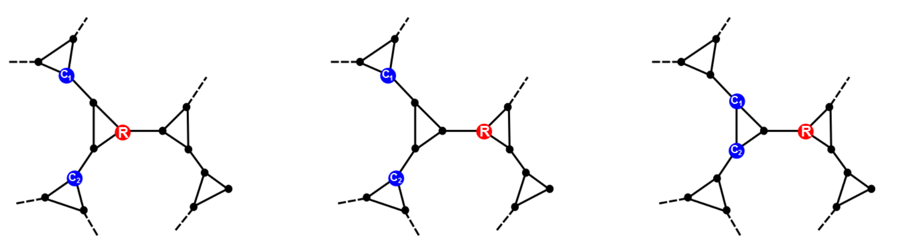
After the cops move they are either one step closer, one step further, or equally as far from the robber, thus their distances from the robber are still "good":

- $d(C_1, R) \geq 1$
- $d(C_2, R) \geq 4$

**Case 2:** $2 \leq d(C_1, R) \leq 4$ and $2 \leq d(C_2, R) \leq 4 \quad \rightarrow \quad$ Robber moves to vertex v which is not $T_{C_1}$ or $T_{C_2}$

1. When v was not on Robber's Triangle, the robber is now further from both cops: $3 \leq d(C_i, v) \leq 5$ (since the truncated girth of the graph is 10, if $d(C_i, v) = 5$ from one side of the 10-cycle the Robber is on, it must also be 5 from the other side). The cops can now move one step closer, one step further, or equally as far from the robber. Therefore, once moved, their distances from the robber are still "good" (see Figure 4):

    - $2 \leq d(C_1, R) \leq 6$
    - $2 \leq d(C_2, R) \leq 6$

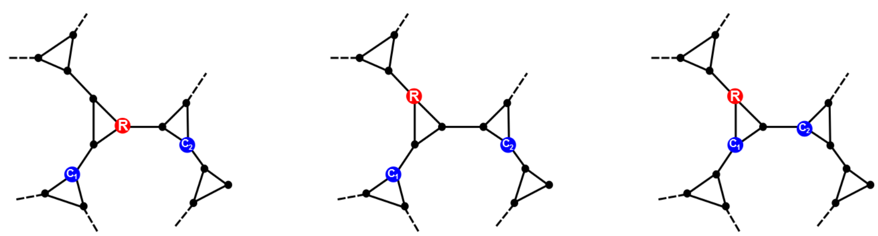



(a) Original position (b) After robber's move (c) After cops' move

Figure 4: Case 2: Part 1

2. When v was on the Robber's Triangle, the robber is now further from cop a and equally as far from cop b as he was at R: either $3 \leq d(C_a, v) \leq 5$ (the cop cannot traverse the robber's large cycle from the opposite

direction, and get within distance 3 because of the truncated girth) or $2 \leq d_(C_b, v) \leq 4$. After the cops move, the distance to v from cop b is $1 \leq d(C_b, v) \leq 5$ where the new $T_{C_b}$ is on the new Robber's triangle. The distance to v from the cop a is $2 \leq d(C_a, v) \leq 6$ where the new $T_{C_a}$ is on the new Robber's triangle. It is possible that cop a could traverse the robber's large cycle from the opposite direction, ending up distance 4 from the cop a, where $T_{C_a}$ is not on the new Robber's triangle, but this is still a "good" position (see Figure 5).



(a) Original position (b) After robber's move (c) After cops' move

Figure 5: Case 2: Part 2

**Case 3:** $d(C_1, R) = 1$ where $C_1$ is not on Robber's Triangle, and $d(C_2, R) \geq 2$ $\rightarrow$ Robber moves to vertex v which is not $T_{C_1}$ or $T_{C_2}$

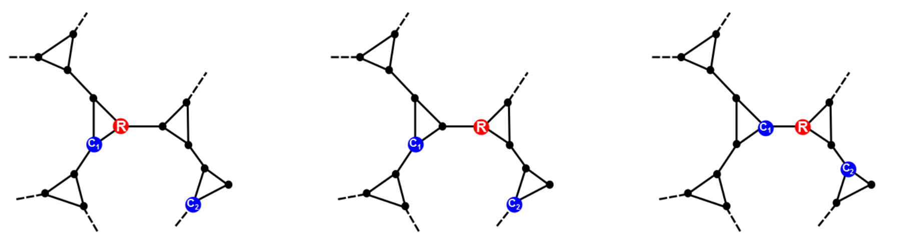
The robber is now further from $C_1$ and the same distance as R was from $C_2$ to $C_2$. After the cops move, the new $T_{C_1}$ and $T_{C_2}$ are both on the Robber's Triangle, satisfying a "good" position, regardless of distance. If $C_2$ tries to traverse the robber's large cycle from the opposite direction, he will be at least distance 4 from the robber where the new $T_{C_2}$ is not on the new Robber's Triangle, which is still a "good" position.

**Case 4:** $d(C_1, R) = 1$ where $C_1$ is on Robber's Triangle, and $d(C_2, R) \geq 4$ $\rightarrow$ Robber moves to vertex v which is not on Robber's Triangle

1. When $T_{C_2}$ was on the Robber's Triangle, the robber moves further from both cops: $d(C_1, v) = 2$ and $d(C_2, v) \geq 5$ ($C_2$ could potentially have a new geodesic around the opposite side of the robber's large cycle such that $d(C_2, v) = 4$ where the new $T_{C_2}$ is on the new Robber's Triangle).

The cops can now move one step closer, one step further, or equally as far from the robber: $1 \leq d(C_1, v) \leq 3$ where the new $T_{C_1}$ is not on the Robber's Triangle and $d(C_2, v) \geq 4$ where $T_{C_2}$ is not on the Robber's triangle or $3 \leq d(C_2, v) \leq 4$ where $T_{C_2}$ is on the new Robber's Triangle. Therefore, once moved, their distances from the robber are still "good".

2. When $T_{C_2}$ was not on the Robber's triangle, the robber moves further from $C_1$ and closer to $C_2$: $d(C_1, v) = 2$ and $d(C_2, v) \geq 3$. After the cops move, $1 \leq d(C_1, v) \leq 3$ where the new $T_{C_1}$ is not on the new Robber's Triangle and $2 \leq d(C_2, v) \geq 4$ where $T_{C_2}$ is on the new Robber's Triangle, thus this new position is a "good" position (see Figure 6).



(a) Original position (b) After robber's move (c) After Cops' move

Figure 6: Case 4: Part 2

**Case 5:** $d(C_1, R) = 1$ where $C_1$ is on Robber's Triangle, and $d(C_2, R) = 1$ where $C_2$ is on Robber's Triangle. $\rightarrow$ Robber moves to vertex v which is not on Robber's Triangle

The robber is now further from $C_1$ and further from $C_2$, where the distance from v to either cop is $d(C_i, v) = 2$. After the cops move, $1 \leq d(C_i, v) \leq 3$ where neither of the new $T_{C_1}$ and $T_{C_2}$ are on the Robber's Triangle, satisfying a "good" position.

**Note:** If there is ever a case where $T_{C_1} = T_{C_2}$, the robber should move according to the rule that fits the case. If the robber has two vertices he could move to that fit the requirements, choose either vertex.

□

# 5 Experimental Mathematics

The development of the faster algorithm for checking cop number is a boon to experimental data on the cop number. While the next section does not contain any mathematical proofs, it does survey possible underlying patterns worthy of further investigation. Some of the most intriguing patterns arise from a type of graph family called an expander family.

## 5.1 Algorithms

Recently a computer algorithm was created that, given a graph $G$ and an integer $k$, checks if $k$ cops can win on $G$ (see Bonato and Nowakowski [2]). That algorithm uses the $k$-fold Strong product of $G$ with itself to track all possible positions of $k$ cops in $G$, and has a worst-case time complexity of $\mathcal{O}(n^{3k+3})$.

The k-th strong product $G^k$ of a graph $G$ is the graph which has as vertex set the $k$-th direct product of the vertex set of $G$, $|V(G)|^k$, and has an edge between any two vertices $(u_0, u_1, u_2, ...u_k)$ and $(v_0, v_1, v_2, ..., v_k)$ if and only if $u_i$ is adjacent to $v_i$ or $u_i = v_i$ for $0 \leq i \leq k$ . Each vertex $v$ in $G^k$ encodes the positions of k cops on $G$, with $v(i)$ encoding the position of cop $i$. However, we see that this way of looking at the cop positions creates redundancy, as the cop-position tuples $(x, y)$ and $(y, x)$ would be considered separate, even though from the point of view of the game, cops are indistinguishable and so having cop 1 on position x and cop 2 on position y is equivalent to having cop 1 in position y and cop 2 in position x.

We resolve this redundancy by considering all cops as indistinct, which we do by creating what we call the k-cop-state graph $G'^k$. This graph has as vertices all k-element sets which allow repetition, which means that the vertices of $G'^k$ allow multiple cops to be on the same vertex, but they do not distinguish between cop positions that are merely permutations of one another. As this set is the the multiset, we know from elementary combinatorics that there are $\frac{(k+|V(G)|-1)!}{k!(|V(G)|-1)!}$ vertices of $G'^k$.

Thus, we have taken an algorithm which requires the construction of a $\mathcal{O}(n^k)$ vertex graph, and reduced it to one that requires the construction of a sub-exponential $\mathcal{O}(\frac{(n+k-1)!}{k!(n-1)!})$ vertex order graph.

## 5.2 Expanders

There exist many types of expander families; that is, infinite sequences of regular graphs characterized by the asymptotic behavior of their isoperimetric constants: they are bounded away from zero. Of particular interest is a 3-regular family $\{H_p\}$ constructed as follows:

- $V(H_p) = \mathbb{Z}_p$ where $p$ is an odd prime.
- $x \in V(H_p)$ is adjacent to $x+1$ and $x-1$.
- $x \neq 0 \in V(H_p)$ is adjacent to $x^{-1}$, its multiplicative inverse. Also, there is a loop at 0.

These rules result in a 3-regular graph with girth at most 7 (since $2^{-1} = \frac{p+1}{2}$ and $(-2)^{-1} = \frac{p-1}{2}$). The improved algorithm applied to the first 23 primes (up to 83) found that in all cases, the cop number of $H_p$ is 2 or 3 (provided the graph has sufficiently many vertices: $H_3$ is trivially copwin). A definite proof strategy has yet to be formulated, and it is certainly possible that some of the larger members of $\{H_p\}$ have arbitrarily high cop number.

Notably, the exact girth of each graph can be computed using elementary algebra over a finite field, which turns into solving a quadratic equation over such a field. For example, let's check a graph for 3-cycles. A 3-cycle occurs when the inverse of an element is 2 greater than it: $x^{-1} = x + 2$. Thus we must solve $x^2 + 2x - 1 = 0$ modulo $p$. All other cycles can be found in a similar way, albeit with more complicated quadratic equations. The fact that the girth is bounded at 7 prevents the use of a result by Frankl [5] which states that if a graph has girth at least $8t - 3$ for some $t \in \mathbb{N}$ and minimum degree above $d$, then $c(G) \geq d^t$.

Another family of expander graphs we studied was the construction found by G. Margulis. The family $\{M_n\}$ is defined as follows:

- $V(M_n) = \mathbb{Z}_n \times \mathbb{Z}_n$ for $n \in \mathbb{N}$.
- The vertex $(x, y)$ is adjacent to: $(x, y + x), (x, y - x), (x + y, y), (x - y, y), (x, y + x + 1), (x, y - x + 1), (x + y + 1, y), (x - y + 1, y)$.

The adjacency relations are not symmetric so the graph can be interpreted as directed. However, we looked viewed each edge as undirected. Such graphs are resistant to computer simulation, as the number of edges increases rapidly.

Here it seems possible that the cop number is unbounded. Exact evaluation of its asymptotic behavior is also difficult.

### 5.3 Smallest $k$-cop-win Graphs

Earlier this year, Beveridge et. al. [1] achieved a proof that the Petersen graph is the smallest 3-cop win graph. Their proof is difficult to generalize, as it relies on a case-by-case breakdown, but it inspires many other intriguing questions. What is the smallest 4-cop win graph? This is an open problem. If such a graph is 4-regular and has girth $\geq 5$, then it must have at least 19 vertices: The unique (4,5)-cage is the Robertson graph, a promising candidate for the smallest such graph. The Robertson graph is at least 4-cop-win by a result of Aigner and Fromme [3]. The algorithm showed it to be exactly 4-cop-win. An upper bound on the smallest $k$-cop win graph of $2^k$ can be found using the method of Petersenication: each iteration of Petersenication doubles the number of vertices and increases the cop number by at least 1. For a prime power $k$, a much better bound of $\sqrt{k}$ can be found by the bipartite graphs that arise from finite projective planes.

## 6 Conclusion

In the future, a deeper study of expander graphs could yield interesting results. Also, notably, the cop number of a polyhedron never decreased with truncation. This could also be a general pattern.

Petersenication yields an upper bound on the smallest $k$-cop-win graph, but the Robertson graph and the projective plane bipartite graphs show that this bounds can be improved considerably.

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[5] Peter Frankl. Cops and robbers in graphs with large girth and Cayley graphs. *Discrete Appl. Math.*, 17(3):301–305, 1987.