Convergence of the Maximum Roots of Fibonacci-Type Functions

Ashley Huhman, Leah Kaisler

August 12, 2013

Abstract

In this paper, we will characterize the nature of the maximum real roots of the Fibonacci-like polynomial sequences given by $G_{k,n}(x) = x^k G_{k,n-1}(x) + G_{k,n-2}(x)$ with $G_{k,0}(x) = -1$, $G_{k,1}(x) = x^{\alpha} - 1$ for $n \geq 2$. Here $k \in (0, 2)$ and $\alpha = 1$ or $\alpha = 2$. Let $f_{k,n}$ denote the maximum real root of $G_{k,n}(x)$. In this paper, we present asymptotic results of of the maximum roots of $G_{k,n}(x)$ when $\alpha = 1$ as $k \to 1$ and $k \to 2$. We will also present similar results when $\alpha = 2$ and $k = 1$.

1 Introduction

The Fibonacci-like polynomial sequence is generated by the second order linear recureence relation

$$G_n(x) = x^k G_{n-1}(x) + G_{n-2}(x),\ n \geq 2$$

with $G_0(x) = a$, $G_1(x) = bx + c$. If $a = b = 1$, and $c = 0$, one gets the classical Fibonacci polynomilal sequence. Bicknell and Hoggatt found explicit forms of the roots for these polynomials in [B73]. Note that, for this case, by setting $x = 1$ one gets the Fibonacci numbers. Many researchers have studied the behavior of the roots of Fibonacci-like polynomials; For $k = 1$, $G_0(x) = -1$, and $G_1(x) = x - 1$, Moore found that the maximum real roots converge to $\frac{3}{2}$ [M94]. Molina and Zeleke found that, when $k = 2$, the maximum real roots converge to $\sqrt{2}$ [M07]. They also proved that, when k is any integer, the maximum real roots converge to some number α_k [M09].

In this paper, we will explore the Fibonacci-type polymnomials under the follwoing cases.

- Part 1: $k \in (0, 2)$, $G_0(x) = -1$, and $G_1(x) = x - 1$
- Part 2: $k = 1$, $G_0(x) = -1$ and $G_1(x) = x^2 - 1$.

2 Fibonacci-Like Polynomials Part I

For $k \in (0,2)$, let $G_{k,0}(x) = -1, G_{k,1}(x) = x - 1$ and $G_{k,n}(x) = x^k G_{k,n-1}(x) + G_{k,n-2}(x)$.

2.1 Technical Results

Lemma 1. *For $n \geq 2$, $G_n(1) = -F_{n-2}$ where F_n denotes the n^{th} Fibonacci number.*

Proof. It is easy to see that $G_2(1) = -1$, $G_3(1) = -1$, and $G_4(1) = -2$. Proceeding by induction, assume that $G_i(1) = -F_{i-2}$ for $2 \leq i \leq n$. Then,

$$G_{n+1}(1) = 1^k G_n(1) + G_{n-1}(1)$$
$$G_{n+1}(1) = -F_{n-2} - F_{n-3}$$
$$F_{n+1}(1) = -F_{n-1}.$$

□

Lemma 2. $G_n(2) > 0$ *for all* $n \geq 1$.

Proof. It is easy to see that $G_2(2) = 2^k(2-1) - 1 > 0$ for $k \in (0,2)$. Assume now $G_i(2) > 0$ for all $2 \leq i \leq n$. Then,

$$G_{n+1}(2) = 2^k G_n(2) + G_{n-1}(2) > 0.$$

□

Lemma 3. $G_n(0) = -1$.

Proof. It's easy to see this for $G_0(0)$, $G_1(0)$, and $G_2(0)$. Assume $G_i(0) = -1$ for all $2 \leq i \leq n$. Then, $G_{n+1}(0) = 0^k G_n(0) + G_{n-1}(0)$ and $G_{n+1}(0) = -1$ by assumption. □

Lemma 4. *For all $k \in (0,2)$, $G_n(x)$ has at least one real root.*

Proof. We know that $G_n(1) < 0$ and $G_n(2) > 0$. By the Intermediate Value Theorem, $G_n(x)$ has a real root on $(1,2)$. □

Lemma 5. $G_n(x) = x^k G_{n-1} + G_{n-2}$ *is increasing on* $[\alpha_{n-1}, \infty)$ *where* α_{n-1} *denotes any root of* G_{n-1} *in* $(1,2)$.

Proof. We will prove this by induction. The first two cases are easy to verify. $G_1(x) = x - 1$ and $G_1'(x) = 1 > 0$ for any x. For $n = 2$,

$$G_2(x) = x^k(x-1) - 1G_2'(x) = kx^{k-1}(x-1) + x^k.$$

Now, since $\alpha_1 = 1$, for any $t > \alpha_1$, we have $G_2'(t) = kt^{k-1}(t-1) + t^k > 0$. Thus $G_2(x)$ is increasing on $[\alpha_{n-1}, \infty)$. Assume that $G_l(x)$ is increasing on $[\alpha_{n-1}, \infty)$

for all $l = 1, 2, 3, ...n-1$. We will show that $G_n(x)$ is increasing on $[\alpha_{n-1}, \infty)$. The derivative of G_n is given by

$$G_n'(x) = kx^{k-1}G_{n-1}(x) + x^k G_{n-1}'(x) + G_{n-2}'(x).$$

and for any $t > \alpha_{n-1}$, we have

$$G_n'(t) = kt^{k-1}G_{n-1}(t) + t^k G_{n-1}'(t) + G_{n-2}'(t) > 0.$$

Note that $G_{n-1}(t) \geq 0$ since $t > \alpha_{n-1}$ and $G_{n-1}(\alpha_{n-1}) = 0$ and $G_{n-1}'(t) > 0$, $G_{n-2}'(t) > 0$ by assumption. Therefore, G_n is increasing on $[\alpha_{n-1}, \infty)$ which implies that G_n has a single real root in the interval $(1, \infty)$. □

The following proof closely follows a proof done in a paper by Molina and Zeleke [M09]. We include it for completeness.

Lemma 6. $G_{n+m}(f_n) = (-1)^{m+1}G_{n-m}(f_n)$.

Proof. We will prove this by induction. When $m = 1$, we have

$$G_{n+1}(f_n) = (f_n)^k G_n(f_n) + G_{n-1}(f_n).$$

Because $G_n(f_n) = 0$, $G_{n+1}(f_n) = G_{n-1}(f_n)$. Also,

$$G_n(f_n) = (f_n)^k G_{n-1}(f_n) + G_{n-2}(f_n)$$

so

$$(f_n)^k G_{n-1}(f_n) = -G_{n-2}(f_n)$$

Now, by assumption,

$$\begin{aligned} G_{n+m}(f_n) &= (f_n)^k G_{n+m-1}(f_n) + G_{n+m-2}(f_n) \\ &= (f_n)^k (-1)^m G_{n-m+1}(f_n) + (-1)^{m-1} G_{n-m+2}(f_n) \end{aligned}$$

Furthermore,

$$G_{n-m+2}(f_n) = (f_n)^k G_{n-m+1}(f_n) + G_{n-m}(f_n)$$

which implies

$$G_{n-m}(f_n) = G_{n-m+2}(f_n) - (f_n)^k G_{n-m+1}(f_n).$$

Therefore,

$$\begin{aligned} \frac{G_{n+m}(f_n)}{G_{n-m}(f_n)} &= \frac{(f_n)^k(-1)^m G_{n-m+1}(f_n) + (-1)^{m-1}G_{n-m+2}(f_n)}{G_{n-m+2}(f_n) - (f_n)^k G_{n-m+1}(f_n)} \\ &= \frac{(-1)^{m-1}(-(f_n)^k G_{n-m+1}(f_n) + G_{n-m+2}(f_n)}{G_{n-m+2}(f_n) - (f_n)^k G_{n-m+1}(f_n)} \\ &= (-1)^{m-1}. \end{aligned}$$

□

2.2 The Odd-Indexed Functions

The proofs of the following lemmas are similar to those in [M09].

Lemma 7. *For all $n \geq 1$, $f_{2n-1} < f_{2n+1}$.*

Proof. We will prove this by induction. Beginning with our base case, $f_1 < f_3$, we know that $f_1 = 1$. Therefore, $G_3(f_1) = G_3(1) = -1$ by Lemma 1. We also know that $G_3(f_3) = 0$ since f_3 denotes the maximum real root of G_3. We see that $G_3(f_1) < G_3(f_3)$. Since $G_3(x)$ is increasing on $[f_3, \infty)$ (Lemma 5), $f_1 < f_3$. Assume now that $f_1 < f_{2i+1} < f_{2n-1}$ for all $2 \leq i \leq n$. Then, since f_{2n-3} is the maximum root of $G_{2n-3}(x)$, $G_{2n-3}(f_{2n-1}) > 0$. However, $G_{2n-3}(f_{2n-1}) = -G_{2n+1}(f_{2n-1})$ (Lemma 6) which implies that $G_{2n+1}(f_{2n-1}) < 0$ and $f_{2n-1} < f_{2n+1}$.

□

2.3 The Even-Indexed Functions

Lemma 8. *For all $n \geq 1$, $f_{2n} < f_{2n+2}$.*

Proof. We show this by induction. We begin with the base case $f_2 > f_4$. By Lemma 6, $G_4(f_2) = -G_0(f_2) = 1 > 0$. Since G_4 is increasing on (f_4, ∞) by Lemma 5, $f_2 > f_4$. Assume now that $f_2 > f_{2i} > f_{2n}$ for all $2 \leq i \leq n$. We know that $G_{2n-2}(f_{2n}) = -G_{2n+2}(f_{2n})$ (Lemma 6). By assumption, $G_{2n-2}(f_{2n}) < 0$, so $G_{2n+2}(f_{2n}) > 0$. This implies that $f_{2n+2} < f_{2n}$. □

2.4 A Special Case

This case simply serves to illustrate the diversity of the recursions of these functions. While many graphs are similar between the two cases, this case happens to differ slightly from those similarities. Here, we examine case one when $k = \frac{1}{2s+1}$ and find that, in this instance, $F_{2n}(x)$ has at least two real roots which is not the case for other values of k. We see an graphical example of this:

Definition 1. *The Lucas sequence is given by $L_n = L_{n-1} + L_{n-2}$ with $L_0 = 2$ and $L_1 = 1$.*

Lemma 9. *For $n \geq 1$ and when $k = \frac{1}{2s+1}$, $G_n(-1) = (-1)^n L_{n-1}$.*

Proof. We easily show the base cases for $n = 1, 2, 3, 4$:

$$G_1(-1) = -1 - 1 = -2$$
$$G_2(-1) = (-1)^{\frac{1}{2s+1}}(-1 - 1) - 1 = 1$$
$$G_3(-1) = (-1)^{\frac{1}{2s+1}}(1) - 2 = -3$$
$$G_4(-1) = (-1)^{\frac{1}{2s+1}}(-3) + 1 = 4.$$

Assume now that $G_i(-1) = (-1)^i L_{i-1}$ for $1 \leq i \leq n$. Then,

$$\begin{aligned} G_{n+1}(-1) &= (-1)^{\frac{1}{2s+1}} G_n(-1) + G_{n-1}(-1) \\ &= -(-1)^n L_{n-1} + (-1)^{n-1} L_{n-2} \\ &= (-1)^{n+1}(L_{n-1} + L_{n-2}) \\ &= (-1)^{n+1} L_n. \end{aligned}$$

□

Lemma 10. *For all $n \geq 1$, $G_{2n}(x)$ has at least two real roots when $k = \frac{1}{2s+1}$.*

Proof. By Lemma 3, $G_{2n}(0) = -1$. By Lemma 9, $G_{2n}(-1) > 0$. It follows that $G_{2n}(0)G_{2n}(-1) < 0$. Therefore, $G_{2n}(x)$ has a real root in $(-1, 0)$ when $k = \frac{1}{2s+1}$. Finally, by Lemma 3, $G_n(x)$ has a maximum root on $(1, 2)$ as well. □

3 Fibonacci-like Polynomials Part II

Let $G_0(x) = -1, G_1(x) = x^2 - 1$ and $G_n(x) = xG_{k,n-1}(x) + G_{k,n-2}(x)$.

3.1 Technical Results

Lemma 11. *For $n \geq 2$, let F_n denote the n^{th} Fibonacci number. Then*

$$G_n(-1) = (-1)^n F_{n-1} \quad (2)$$

$$G_n(1) = -F_{n-1} \quad (3)$$

$$F_n(2) = 2G_{n-1} + G_{n-2} \quad (4)$$

Proof. The statements in Lemma 11 follow from induction on n:

$$G_n(-1) = (-1)F_{n-1}(-1) + F_{n-2}(-1) \quad (2)$$

$$G_{n+1}(-1) = (-1)^n F_{n-1} - 1^{n-1} F_{n-2}$$

$$G_n(1) = (1)F_{n-1}(1) + F_{n-2}(1) \quad (3)$$

$$G_{n+1}(1) = -F_{n-1} - F_{n-2}$$

$$G_n(2) = (2)F_{n-1}(2) + F_{n-2}(2) \quad (4)$$

$$G_{n+1}(2) = 2F_{n-1} + 3F_{n-2} + F_{n-3}$$

□

Lemma 12. *For $n \geq 2$, $a_n \in (1,2)$ where a_n denotes a real root for this recursion.*

Proof. From the above lemma, we see that $G_n(1) < 0$ and $G_n(2) > 0$. By the Intermediate Value Theorem, $a_n \in (1,2)$. □

Remark 1. *Using a similar proof as in to Lemma 5, we can show that $a_n = f_{1,n}$, where $f_{1,n}$ is the maximum real root of G_n.*

3.2 The Odd-Indexed Functions

The proofs of the following lemmas are similar to that from a paper by Molina and Zeleke [M09].

Lemma 13. *The odd indexed polynomial's maximum roots form an increasing sequence $f_1 < f_3 < f_5 < f_7 < \cdots < f_{2n+1}$.*

Proof. By induction on n we can prove that the maximum real roots are increasing. We can easily see that $f_1 = 1$ and $f_3 \approx 1.2207$, so $f_1 < f_3$. We now assume that $f_1 < f_3 < \cdots < f_{2n-1}$. Since f_{2n-3} is the maximum root of G_{2n-3} and $f_{2n-3} < f_{2n-1}$ by assumption, we can conclude that $G_{2n-3}(f_{2n-1}) > 0$. By Lemma 6, $G_{2n-3}(f_{2n-1}) = -G_{2n+1}(f_{2n-1})$ and this means that $G_{2n+1}(f_{2n-1}) < 0$. Therefore, $f_{2n+1} > f_{2n-1}$. □

3.3 The Even-Indexed Functions

Lemma 14. *The even indexed polynomial's maximum roots are a decreasing sequence, $f_2 > f_4 > \cdots > f_{2n}$.*

Proof. Since $f_2 \approx 1.32472$ and $f_4 \approx 1.25461$, we can proceed by induction. By Lemma 6 and using the same method as the previous lemma, $G_{2n+2}(f_{2n}) = -G_{2n-2}(f_{2n})$. Then, $G_{2n-2}(f_{2n}) > 0$ by assumption, so $G_{2n+2}(f_{2n}) < 0$. Therefore, $f_{2n} > f_{2n+2}$. □

4 Main Results

Lemma 15. *The sequence $f_{k,2n+1}$ is bounded from above by β_k (β_k is defined below) for the recursion $G_{k,n}(x) = x^k G_{k,n-1}(x) + G_{k,n-2}(x),\ k \in (0,2)$, $G_{k,0}(x) = -1$, and $G_{k,1}(x) = x - 1$.*

Proof. As shown in [M09], if $G_{k,2}(x) = -(1-x)^2$, then $G_{k,n}(\beta_k) = -(1-\beta_k)^n$ for all n where β_k is a root of $T_k(x) = x^k - x^{k-1} + x - 2$. We can see that $T_k(1) = -1 < 0$ and $T_k(2) = 2^k - 2^{k-1} = 2^{k-1}(2-1) > 0$. By the Intermediate Value Theorem, there exists a root between 1 and 2. This implies that there exists $d \in (1,2)$ such that $T_k(d) = 0$. Thus, $\beta_k \geq d > 1$. Therefore, $1 - \beta_k < 0$ and $G_{k,2n+1}(\beta_k) = -(1-\beta_k)^{2n+1} > 0$. Also, $G'_{k,2n+1}(x) > 0$ for all $x \geq f_{k,2n+1}$ (Lemma 5). Thus, $f'_{k,2n+1} < \beta_k$ since $G_{2n+1}(f_{2n+1}) = 0$. This implies that $f_1 < f_3 < \cdots < f_{2n+1} < \beta_k$ which implies that the limit of f_{2n+1} exists as $n \to \infty$ and $\lim_{n\to\infty} f_{2n+1} \leq \beta_k$. □

Remark 2. *We use a similar proof to show $\beta_k < f_{2n} < ... < f_4 < f_2$.*

Remark 3. *Using similar arguments, one can show that the maximum real root $f_{1,n}(x)$ is bounded by 2 for the recursion $G_{1,n}(x) = xF_{1,n-1}(x) + G_{1,n-2}(x)$, $G_{1,0}(x) = -1$, and $G_{1,1}(x) = x - 1$.*

5 Acknowledgements

The authors would like to thank the National Security Agency and the National Science Foundation for funding this program (project sponsored by the National Security Agency under Grant Number H98230-13-1-0259; project sponsored by the National Science Foundation under Grant Number DMS 1062817). They would also like to thank Michigan State University and Lyman Briggs College for hosting the REU. Finally, they would like to thank Dr. Aklilu Zeleke, Rani Satyam, Justin Droba, and Richard Shadrach for all of their assistance.

6 References

[B73] Bicknell, M. and Hoggatt Jr., V.E., *Roots of Fibonacci Polynomials,* The Fibonacci Quarterly, **11.3,** (1973), 271-274.

[M09] Molina, R. and Zeleke, A., *Generalizing Results on the Convergence of the Maximum Roots of Fibonacci-Type Polynomials,* Congressus Numerantium, **195** (2009), pp 95-104.

[M07] Molina, R. and Zeleke, A.,*On the Convergence of the Maximum Roots of a Fibonacci-Type Polynomial Sequence,* Congressus Numerantium **184** (2007), pp. 121-128.

[M94] Moore, G.,*The Limit of the Golden Numbers is $\frac{3}{2}$,* The Fibonacci Quarterly, **32.3** (1994), 211-217.