Remarks on the Properties of a Quasi-Fibonacci-like Polynomial Sequence

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Abstract

Consider the Quasi-Fibonacci-like Polynomial Sequence given by $F_0 = -1, F_1 = x - 1$ and for $n \geq 2$, $F_n = F_{n-1} + x^2 F_{n-2}$. Denote the maximum root of F_n by g_n. In this article, we will analyze the existence and nonexistence of g_n as well as study the behavior of the sequence $\{g_n\}$. We will prove all but one of the roots are irrational and the sequence of the maximum odd-indexed roots are monotonically increasing.

1 Introduction

Consider the well-studied Fibonacci Polynomial Sequence,

$$F_n = xF_{n-1} + F_{n-2}, \quad \text{for} \quad n \geq 2 \quad \text{with} \quad F_0 = 0, F_1 = 1.$$

Many results are known about this polynomial sequence. It is known that $F_n(1)$ is the n^{th} Fibonacci number. Hogatt and Bicknell [HB] also gave an explicit form for the zeros of these polynomials.

Further work includes Molina and Zeleke [MZ] generalizing of the initial conditions and exploring the recursion,

$$F_n = x^k F_{n-1} + F_{n-2},$$

now known as the Fibonacci-like polynomials. They made a number of discoveries about the asymptotic behavior of the roots of these polynomials.

This inspired further work by Brandon Alberts in 2011. Alberts studied the Quasi-Fibonacci polynomials. These are polynomials defined by the following recursion:

$$F^q_n = F^q_{n-1} + x^k F^q_{n-2}, \quad \text{for} \quad n \geq 2 \quad \text{with} \quad F^q_0 = -1, F^q_1 = x - 1, \quad \text{where} \quad k = 1.$$

He found a number of interesting results including the existence of all roots and convergence of all roots to the same value, namely 2. He found that the roots of the even-indexed polynomials converged from below and the roots of the odd-indexed polynomials converged from above.

In this paper, we study a modified recursion, a Quasi-Fibonacci-Like polynomial sequence, where $k = 2$. Specifically, this recursion is as follows:

$$F^q_n = F^q_{n-1} + x^2 F^q_{n-2}, \quad \text{for} \quad n \geq 2 \quad \text{with} \quad F^q_0 = -1, F^q_1 = x - 1.$$

We will explore the existence and nonexistence of the roots, as well as their behavior as a sequence. We will also numerically and computationally examine the asymptotic behavior of these roots in addition to showing all but one root are irrational. We denote the maximum root of F_n^q as g_n, and for the sake of simplicity, we suppress the q superscript for the rest of this paper.

2 Formulas and technical results

The formulas we will use throughout this paper are as follows:

Lemma 1.

$$F_n(x) = \sum_{i=0}^{\infty} \left[\tbinom{n-i-1}{i} x^{2i+1} - \tbinom{n-i}{i} x^{2i} \right] \tag{1}$$

$$F_{2n}(x) = -\left[\sum_{i=0}^{n} \tbinom{2n-i}{i} x^{2i} \right] + \left[\sum_{i=0}^{n-1} \tbinom{2n-i-1}{i} x^{2i+1} \right] \tag{2}$$

$$F_{2n+1}(x) = \sum_{i=0}^{n} \left[\tbinom{2n-i}{i} x^{2i+1} - \tbinom{2n+1-i}{i} x^{2i} \right] \tag{3}$$

$$F_n(x) = \frac{(x-1+\lambda_-)\lambda_+^n - (x-1+\lambda_+)\lambda_-^n}{\lambda_+ - \lambda_-}, \ \textit{where } \lambda_\pm = \frac{1 \pm \sqrt{1+4x^2}}{2}. \tag{4}$$

$$F_n(x) = (1+2x^2)F_{n-2}(x) - x^4 F_{n-4}(x) \tag{5}$$

Remark. *Formula (4) is known at the Binet Form.*

Proof. **Formula (1)** We proceed by induction. We begin by showing the base cases.

Case 1: $n = 1$

$$\begin{aligned}
&\sum_{i=0}^{\infty} \left[\binom{1-i-1}{i-1}(-1) + \binom{1-i-1}{i}(x-1) \right] x^{2i} \\
&= \left[\binom{0}{-1}(-1) + \binom{0}{0}(x-1) \right] x^0 + \left[\binom{-1}{0}(-1) + \binom{-1}{1}(x-1) \right] x^2 + \dots \\
&= [0 \cdot (-1) + 1 \cdot (x-1)] \cdot 1 + 0 + 0 + \dots \\
&= x - 1 = F_1(x)
\end{aligned}$$

Case 2: $n = 2$

$$\begin{aligned}
&\sum_{i=0}^{\infty} \left[\binom{2-i-1}{i-1}(-1) + \binom{2-i-1}{i}(x-1) \right] x^{2i} \\
&= \left[\binom{1}{-1}(-1) + \binom{1}{0}(x-1) \right] x^0 + \left[\binom{0}{0}(-1) + \binom{0}{1}(x-1) \right] x^2 + \dots \\
&= [0 \cdot (-1) + 1 \cdot (x-1)] \cdot 1 + [1 \cdot (-1) + 0 \cdot (x-1)] x^2 + 0 + \dots \\
&= -x^2 + x - 1 = F_2(x)
\end{aligned}$$

We now show the inductive step. Suppose this identity holds for all $n < m$. Then

$$\begin{aligned}
F_m(x) &= F_{m-1}(x) + x^2 F_{m-2}(x) \\
&= \sum_{i=0}^{\infty} \left[\binom{m-i-2}{i-1}(-1) + \binom{m-i-2}{i}(x-1) \right] x^{2i} \\
&\quad + x^2 \sum_{i=0}^{\infty} \left[\binom{m-i-3}{i-1}(-1) + \binom{m-i-3}{i}(x-1) \right] x^{2i} \\
&= \sum_{i=0}^{\infty} \left[\binom{m-i-2}{i-1}(-1) + \binom{m-i-2}{i}(x-1) \right] x^{2i} \\
&\quad + \sum_{j=1}^{\infty} \left[\binom{m-j-2}{j-2}(-1) + \binom{m-j-2}{j-1}(x-1) \right] x^{2j}.
\end{aligned}$$

Reindexing and combining terms we have

$$\begin{aligned}
F_m(x) &= \sum_{i=0}^{\infty} \left[\left[\tbinom{m-i-2}{i-1} + \tbinom{m-i-2}{i-2} \right](-1) + \left[\tbinom{m-i-2}{i} + \tbinom{m-i-2}{i-1} \right](x-1) \right] x^{2i} \\
&= \sum_{i=0}^{\infty} \left[\tbinom{m-i-1}{i-1}(-1) + \tbinom{m-i-1}{i}(x-1) \right] x^{2i}
\end{aligned}$$

giving Formula (1).

Formulas (2) and (3) follow directly from Formula (1).

Formula (4): This is the Binet Form for this recursion. We proceed by using the standard method of obtaining a Binet Formula. We establish the following system of equations:

$$\begin{cases} c_1\lambda_+^0 + c_2\lambda_-^0 = F_0(x) \\ c_1\lambda_+ + c_2\lambda_- = F_1(x) \end{cases}$$

where λ_+ and λ_- are the solutions to the equation $\lambda^2 = \lambda + x^2$. From this quadratic equation, we find λ_+ and λ_- to be as in Formula (4). Solving the system above, we find

$$c_1 = \frac{x - 1 + \lambda_-}{\lambda_+ - \lambda_-} \quad \text{and} \quad c_2 = \frac{-(x - 1 + \lambda_+)}{\lambda_+ - \lambda_-}.$$

Formula (5): This follows from direct manipulation of the recursion. Indeed we have

$$\begin{aligned}
F_n(x) &= F_{n-1}(x) + x^2 F_{n-2}(x) \\
&= (1 + 2x^2)F_{n-2} - x^2 F_{n-2} + x^2 F_{n-3} \\
&= (1 + 2x^2)F_{n-2} - x^4 F_{n-4}.
\end{aligned}$$

□

Lemma 2. *For all $x < 0$, we have $0 > F_n(x) > F_{n+1}(x)$ for all n.*

Proof. When $x < 0$, $F_1(x) = x - 1 < -1 = F_0(x) < 0$. Furthermore, supposing this statement holds up to F_{n-1}, we see that since $F_{n-2}(x) < 0$, $F_n(x) = F_{n-1}(x) + x^2 F_{n-2}(x) < F_{n-1}(x)$. □

Lemma 3. *$F_n(0) = -1$ for all n.*

Proof. $F_n(0)$ will be the constant term of F_n; according to our Formula (1), this will be $-\binom{n-0}{0} = -1$ □

Lemma 4. *$\lim_{x\to\infty} F_{2n+1}(x) = \infty$ for all n.*

Proof. First we show this is satisfied for initial values of n. Proceeding by induction we have the base cases

$$\lim_{x\to\infty} F_1 = \lim_{x\to\infty} x - 1 = \infty$$

and

$$\lim_{x\to\infty} F_3 = \lim_{x\to\infty} x^3 - 2x^2 + x - 1 = \infty.$$

Suppose we've shown this for $F_1, F_3, ..., F_{2n-1}$. By using Formula (3), we have

$$\lim_{x\to\infty} F_{2n+1}(x) = \lim_{x\to\infty} \sum_{i=0}^{n} \left[-\binom{2n-i}{i-1} + \binom{2n-i}{i}(x-1)\right] x^{2i}.$$

Since the end behavior of a polynomial is determined by the sign of the coefficient on the highest degree, it is sufficient to show that this coefficient is positive.

Note that $\binom{2n-i}{i} = 0$ when $2n - i > i$. However the leading coefficient must be nonzero, so $2n - i \leq i$. This implies $n \leq i$. Examining the term with the highest degree, we see

$$\begin{aligned}
&\left[-\binom{2n-(n)}{(n)-1} + \binom{2n-(n)}{n}(x-1)\right] x^{2(n)} \\
&= \left[-\binom{n}{n-1} + \binom{n}{n}(x-1)\right] x^{2n} \\
&= \left[-\binom{n}{n-1} + x - 1\right] x^{2n} \\
&= -\binom{n}{n-1} x^{2n} + x^{2n+1} - x^{2n}.
\end{aligned}$$

Thus the coefficient of the term with highest degree, namely x^{2n+1}, is positive and therefore

$$\lim_{x\to\infty} F_{2n+1} = \infty.$$

□

Lemma 5. *$\lim_{x\to\infty} F_{2n}(x) = -\infty$, for all n.*

Proof. Lemma 5 can be proven using similar methods to Lemma 4. □

3 Existence and nonexistence of the roots

Lemma 6. *For any x and any $n \geq 1$, $0 > F_{2n}(x) \geq F_{2n+2}(x)$. In particular, $F_{2n}(x)$ has no real roots.*

Proof. When $x \leq 0$, this statement follows directly from Lemmas 2 and 3. So let $x > 0$. We now proceed by induction, starting with the base cases. Notice that

$$F_2(x) = -x^2 + x - 1 < 0$$
and
$$F_2(x) - F_4(x) = x^4 - 2x^3 + 2x^2 = x^2\left((x-1)^2 + 1\right) > 0,$$

so $0 > F_2(x) > F_4(x)$ for all real values of x.
We now show the inductive step. Suppose now that we have shown this property through F_{2n-2}.

Case 1: $x \leq \sqrt{2}$. Then

$$2x^2 F_{2n-2}(x) \leq x^4 F_{2n-2}(x) \leq x^4 F_{2n-4}(x).$$

From Formula 5, we have

$$F_{2n}(x) = (1 + 2x^2)F_{2n-2}(x) - x^4 F_{2n-4}(x) \leq F_{2n-2}(x).$$

Case 2: $x > \sqrt{2} > 0$. Recall $c_1 = \frac{x-1+\lambda_-}{\lambda_+ - \lambda_-}$ and $c_2 = \frac{-(x-1+\lambda_+)}{\lambda_+ - \lambda_-}$.

$$\begin{aligned} 0 &< x \\ 0 &> -4x \\ 4x^2 - 4x + 1 &< 1 + 4x^2 \\ (2x-1)^2 &< 1 + 4x^2 \end{aligned}$$

Taking the primary root of both sides provides,

$$\begin{aligned} (2x-1) - \sqrt{1+4x^2} &< 0 \\ c_1 &< 0. \end{aligned}$$

A similar proof shows $c_2 < 0$. Now we will show $1 - \lambda_+ - x^2 < 0$.

$$\begin{aligned} \sqrt{2} &< x \\ 0 &< x^2 - 2 \\ 0 &< 4x^4 - 8x^2 \\ 1 + 4x^2 &< 4x^4 - 4x^2 + 1 \end{aligned}$$

Taking the primary root of both sides provides

$$\begin{aligned} 1 - 2x^2 + \sqrt{1+4x^2} &< 0 \\ 1 - \lambda_+ - x^2 &< 0. \end{aligned}$$

A similar proof shows $1 - \lambda_- - x^2 < 0$.

Since $c_1, c_2, (1 - \lambda_+ - x^2)$, and $(1 - \lambda_- - x^2)$ are all negative, from Formula 4 giving the Binet form of F_n, we have

$$\begin{aligned} F_{2n-2} - F_{2n} &= c_1\lambda_+^{2n-2} + c_2\lambda_-^{2n-2} - c_1\lambda_+^{2n} - c_2\lambda_-^{2n} \\ &= c_1\lambda_+^{2n-2}(1 - \lambda_+ - x^2) + c_2\lambda_-^{2n-2}(1 - \lambda_- - x^2) \\ &> 0 \end{aligned}$$

Thus $F_{2n-2} > F_{2n}$ so that for all x, $F_{2n}(x)$ is negative and therefore has no real roots. □

Lemma 7. *F_{2n+1} has at least one real root.*

Proof. Notice that the leading coefficient of F_{2n+1} is positive and $F_{2n+1}(0) = -1$.
By the intermediate value theorem, F_{2n+1} has at least one real root. □

4 Main Results

For the remainder of this paper, let g_n denote the maximum real root of F_n.

Theorem 1. $1 = g_1 < g_3 < \cdots < g_{2n+1}$ *for all n.*

Proof. We proceed by induction. Starting with the base cases direct computation shows $1 = g_1 < g_3$. We now show the induction step. Suppose we have shown this through g_{2n-1}. Then since g_{2n-3} is a maximum root and $\lim_{x\to\infty} F_{2n-3}(x) = +\infty$, we note that $F_{2n-3}(g_{2n-1}) > 0$. So

$$\begin{aligned} F_{2n+1}(g_{2n-1}) &= (1 + 2g_{2n-1}^2) \cdot F_{2n-1}(g_{2n-1}) - g_{2n-1}^4 \cdot F_{2n-3}(g_{2n-1}) \\ &= -g_{2n-1}^4 \cdot F_{2n-3}(g_{2n-1}) < 0. \end{aligned}$$

Thus, since $\lim\limits_{x\to\infty} F_{2n+1}(x) = +\infty$, F_{2n+1} must have a root $g_{2n+1} > g_{2n-1}$. □

Theorem 2. *The roots of F_{2n+1} are irrational for $n > 0$.*

Proof. If there is some rational number r which is a root of F_{2n+1} then, by the Rational Root Theorem and Formula 3,

$$r = \pm\frac{\left[-\binom{2n-n}{n-1} + \binom{2n-n}{n}\right]}{\left[-\binom{2n}{-1} + \binom{2n}{0}\right]}$$

$$= \pm(-n + 1).$$

Case 1: $r = -n + 1$. By Theorem 1, $g_{2n+1} > 1$. However for all n, we have $r \leq 1$. So $-n + 1$ cannot be a root.

Case 2: $r = n-1$. We have shown this cannot be a root for $n < 62$ by direct computation. Through manipulation of Formula (3) we have

$$F_{2n+1}(x) = \sum_{i=0}^{n+1} \left[-\binom{2n-i}{i-1} + \binom{2n-i}{i}(x-1) \right] x^{2i}.$$

When substituting r into the expression above for x, it is easy to show that only the coefficient of the last term is negative. It is given by

$$(-2)(n-1)^{2n}.$$

Finding the ratio between the original binomial coefficients provides,

$$\binom{2n-i}{i-1} = \binom{2n-i}{i} \frac{i}{2n-2i+1}.$$

Since the last term is the only negative term, it is sufficient to show that the third to last term is larger. Using $i = n-2$ with $n \geq 62$ gives,

$$\left(\frac{n+2}{n-1}\right)\left(\frac{n+1}{n-1}\right)\left(\frac{n}{n-1}\right)\left(\frac{4n}{5} - \frac{8}{5}\right) \geq 48.$$

$$\left(\frac{(n-2)(n+1)(n)(n-1)}{24}\right)\left(\frac{4n}{5} - \frac{8}{5}\right) \geq 2(n-1)^4$$

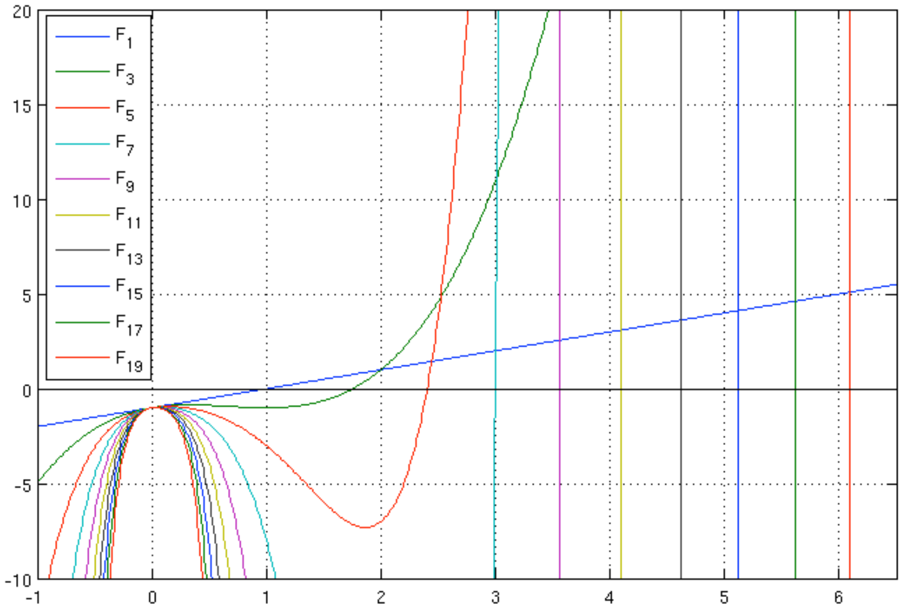
$$\binom{2n-(n-2)}{n-2}\left(\frac{-(n-2)}{2n-2(n-2)+1} + n - 2\right)(n-1)^{2(n-2)} \geq 2(n-1)^{2n}$$

where the left hand side is the third to last term in the above summation. Therefore, g_{2n+1} is irrational for all $n > 0$. □

5 Numerical evidence

Our research has also suggested certain other results. One such possibility is that the odd-indexed polynomials, F_{2n+1}, have exactly one real root.

As the following graph shows, the first few odd-indexed polynomials have exactly one real root.



Thus we have constructed the following lemma and conjecture.

Lemma 8. *F_{2n+1} has no real roots on $(-\infty, g_{2n-1}] \cup (g_{2n+1}, \infty)$.*

Proof. This is obvious for $F_1(x) = x - 1$. Suppose we have shown it through F_{2n-1}. When $x > g_{2n+1}$, this follows directly from maximality of g_{2n+1}. Therefore, let $x \leq g_{2n-1}$. Recall that $F_{2n}(x) < 0$ and note that $F_{2n-1}(x) \leq 0$. Then

$$F_{2n+1}(x) = F_{2n}(x) + x^2 F_{2n-1}(x) < 0$$

□

The previous lemma does not exclude the possibility of a root existing on the interval (g_{2n-1}, g_{2n+1}), providing the following conjecture formed through observation.

Conjecture 1. *F_{2n+1} has no real roots on (g_{2n-1}, g_{2n+1}).*

This has been confirmed by direct calculation through F_{599}, meaning all of these polynomials have exactly one real root. It is our hope that future work may be able to formally prove this for all odd-indexed F_n.

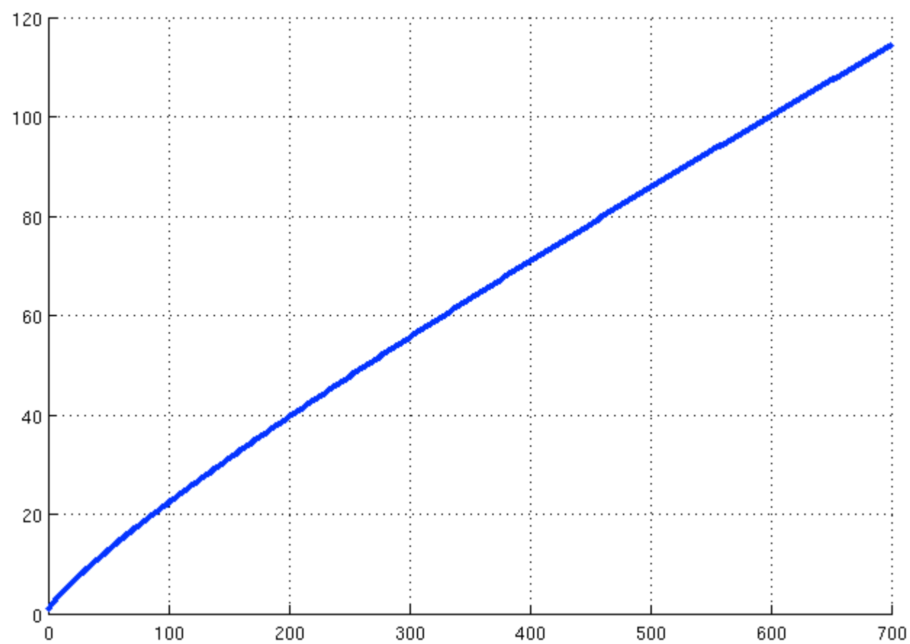
Our work has also led us to believe the following conjecture:

Conjecture 2. *The sequence $\{g_{2n+1}\}$ is unbounded.*

The roots do not appear to asymptotically approach any number through g_{599}, selected roots are shown below.

n	g_n
1	1
3	1.755
5	2.402
13	4.616
29	8.390
101	22.544
233	44.969
419	73.762
599	100.05

Graphically, we can also see that they do not appear to asymptotically approach any number. After multiple regression analyses, we can say that it appears to grow slightly faster than logarithmically.



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7 References

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