Location of Roots for Recursively Defined Sequences

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Abstract
Consider the recursively defined sequences

\[ F_n(x) = g(x)F_{n-1}(x) + h(x)F_{n-2}(x) \]

with initial conditions \( F_0 \) and \( F_1 \). In this article, we introduce a new way to bound the roots of \( F_n(x) \) and show how this new method can be used with respect to the Fibonacci-type polynomials. Furthermore, we compare this root bounding method with previously used methods involving Geršchgorin’s Circle Theorem.

1 Introduction

Consider a Fibonacci type polynomial sequence defined by

\[ G_n(x) = x^kG_{n-1}(x) + G_{n-2}(x), \quad n \geq 2 \]

with given initial conditions \( G_0(x) \) and \( G_1(x) \). When \( G_0(x) = 1, G_1(x) = x \) and \( k = 1 \), one gets the Fibonacci polynomial sequence. For \( G_0(x) = 2, G_1(x) = x \) and \( k = 1 \) one gets the Lucas polynomial sequence. Hogatt and Bicknell [HB], give explicit forms for the roots of the Fibonacci and Lucas polynomials; however, finding explicit forms for the roots of other polynomials sequences has been a challenge. Despite this, Moore [Moo] and Prodinger [Pro] studied the asymptotic behavior of the maximal roots of \( G_n(x) \) with \( G_0(x) = -1, G_1(x) = x - 1 \) and \( k = 1 \). Then Yu, Wang and He [YWH] generalized Moore’s result for \( G_0(x) = a, \)
\( G_1(x) = x + a \) and \( k = 1 \) when \( a \) is a negative integer. Additionally, Molina and Zeleke found similar results for \( k \geq 2 \) in [MZ1] and [MZ2].

Regarding the boundedness of the roots for \( G_n(x) \): Ricci [Ric], Mátyás [Mát], and Wang and He [WH] examined boundedness of the roots for \( G_n(x) \). With \( G_0(x) = a, G_1(x) = x + b \), Wang and He [WH] proved the roots are bounded by \( 1 + \max \{|a|, |b|\} \) which generalized the results of Ricci [Ric] and Mátyás [Mát].

In this article we develop a new method to show the boundedness of the roots for a more general recursions. We then show how our method compares with the method used by Wang and He [WH] involving Gerschgorin’s Circle Theorem, and finally, we introduce a sufficient condition for a recursion to have bounded roots.

## 2 Bounding Roots of Recursive Functions

Consider a function, \( F(x) \), that can be represented by the determinant of an \( n \times n \) matrix, \( A_n \). Denote the entry in the \( i \)th row, and \( j \)th column of \( A_n \) by \( f_{ij}(x) \) where \( f_{ij}(x) \) is defined for all \( x \). Then we have the following theorem:

**Theorem 2.1.** Each zero of \( F(x) \) satisfies at least one of the \( n \) inequalities given by:

\[
|f_{ii}(x)| \leq \sum_{1 \leq j \leq n, j \neq i} |f_{ij}(x)|, \quad i = 1, 2, ..., n.
\]

*Proof.* We will prove this by contradiction. Assume the opposite, that some root, \( r \), of \( F(x) \) satisfies:

\[
|f_{ii}(r)| > \sum_{1 \leq j \leq n, j \neq i} |f_{ij}(r)|, \quad i = 1, 2, ..., n.
\]

However if this is the case, then \( A_n \) by definition is a strictly diagonally dominant matrix. Since \( |A_n| = F(r) = 0 \), a contradiction occurs because all strictly diagonally dominant matrices are non-singular [HJ, Theorem 6.1.10].

\[\square\]

Additionally, since the determinant of a square matrix is equal to the determinant of its transpose we have the following corollary:
Corollary 2.2. Each zero of $F(x)$ also satisfies at least one of the $n$ equalities given by:

$$|f_{jj}(x)| \leq \sum_{1 \leq i \leq n \atop i \neq j} |f_{ij}(x)|, \quad j = 1, 2, \ldots, n.$$ 

2.1 Comparison with the Circle Theorem

As shown in [WH], one can use Geršchgorin’s Circle Theorem to bound the eigenvalues of a scalar, square matrix and therefore bound the roots of the associated characteristic polynomial. In their paper, they find bounds for the zeros of a Fibonacci-type polynomial which exactly agrees with the predictions of Theorem 2.1. However, using Geršchgorin’s Circle theorem involves unnecessary steps and also has limitations, especially when the problem of finding a corresponding scalar matrix is difficult. Consider the following example:

$$G_n(x) = x^2G_{n-1}(x) + G_{n-2}(x); \quad G_0(x) = 1, \ G_1(x) = x.$$ 

It is easily checked by induction that

$$G_n(x) = |B| = \begin{vmatrix} x & -1 & \cdot & \cdot & \cdot \\ 1 & x^2 & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & x^2 & \cdot & \cdot & -1 \end{vmatrix}.$$ 

If one proceeds with the aim to use Geršchgorin’s Circle Theorem, they will soon run into difficulties since $G_n(x)$ is the characteristic polynomial of $C$ if

$$G_n(x) = |B| = \det(\lambda I - C).$$

This is problematic because no substitution for $\lambda$ will make $C$ a scalar matrix, and the theorem cannot be applied. However given $G_n(x) = |B|$, one can directly use Theorem 2.1 to obtain that the zeros of $G_n(x)$ satisfy at least one of the following inequalities:

$$|x| \leq 1; \ |x^2| \leq 2; \ |x^3| \leq 1.$$ 

Therefore all the roots satisfy

$$|x| \leq \sqrt{2}.$$
3 General Recursions

For a general reccurance given by

\[ F_n(x) = g(x)F_{n-1}(x) + h(x)F_{n-2}(x), \]

with initial conditions \( F_0(x) \) and \( F_1(x) \), we can show by induction that \( F_n(x) \) is
the determinant of the \( n \times n \) matrix

\[
\begin{vmatrix}
F_1(x) & -F_0(x) \\
h(x) & g(x) & -1 \\
& h(x) & g(x) & \ddots \\
& & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots & -1 \\
& & & & & h(x) & g(x)
\end{vmatrix}
\]

Using this determinant representation of \( F_n(x) \) and Theorem 2.1 we can see that
all the roots of \( F_n(x) \) satisfy at least one of the following inequalities.

\[ |F_1(x)| \leq |F_0(x)| \quad |g(x)| \leq |h(x)| + 1 \]

Furthermore, we can use these inequalities to obtain a sufficient condition for
boundedness of the roots for \( F_n(x) \).

**Theorem 3.1.** Let \( F_n(x) \) be a recursion with initial conditions \( F_0(x) \) and \( F_1(x) \),
and for all \( n \geq 2 \)

\[ F_n(x) = g(x)F_{n-1}(x) + h(x)F_{n-2}. \]

Then if the values of \( x \) that satisfy either \( |F_1(x)| \leq |F_0(x)| \) or \( |g(x)| \leq |h(x)| + 1 \)
are bounded in a region in the complex plane, then the roots of \( F_n(x) \) are bounded
and are located in that region.

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References


