Location of Roots for Recursively Defined Sequences

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Abstract

Consider the recursively defined sequences

$$F_n(x) = g(x)F_{n-1}(x) + h(x)F_{n-2}(x)$$

with initial conditions F_0 and F_1. In this article, we introduce a new way to bound the roots of $F_n(x)$ and show how this new method can be used with respect to the Fibonacci-type polynomials. Furthermore, we compare this root bounding method with previously used methods involving Gerŝchgorin's Circle Theorem.

1 Introduction

Consider a Fibonacci type polynomial sequence defined by

$$G_n(x) = x^k G_{n-1}(x) + G_{n-2}(x), n \geq 2$$

with given initial conditions $G_0(x)$ and $G_1(x)$. When $G_0(x) = 1$, $G_1(x) = x$ and $k = 1$, one gets the Fibonacci polynomial sequence. For $G_0(x) = 2$, $G_1(x) = x$ and $k = 1$ one gets the Lucas polynomial sequence. Hogatt and Bicknell [HB], give explicit forms for the roots of the Fibonacci and Lucas polynomials; however, finding explicit forms for the roots of other polynomials sequences has been a challenge. Despite this, Moore [Moo] and Prodinger [Pro] studied the asymptotic behavior of the maximal roots of $G_n(x)$ with $G_0(x) = -1$, $G_1(x) = x - 1$ and $k = 1$. Then Yu, Wang and He [YWH] generalized Moore's result for $G_0(x) = a$,

$G_1(x) = x + a$ and $k = 1$ when a is a negative integer. Additionally, Molina and Zeleke found similar results for $k \geq 2$ in [MZ1] and [MZ2].

Regarding the boundedness of the roots for $G_n(x)$: Ricci [Ric], Mátyás [Mát], and Wang and He [WH] examined boundedness of the roots for $G_n(x)$. With $G_0(x) = a$, $G_1(x) = x + b$, Wang and He [WH] proved the roots are bounded by $1 + \max\{|a|, |b|\}$ which generalized the results of Ricci [Ric] and Mátyás [Mát].

In this article we develop a new method to show the boundedness of the roots for a more general recursions. We then show how our method compares with the method used by Wang and He [WH] involving Gerŝchgorin's Circle Theorem, and finally, we introduce a suffient condition for a recursion to have bounded roots.

2 Bounding Roots of Recursive Functions

Consinder a function, $F(x)$, that can be represented by the determinant of an $n \times n$ matrix, A_n. Denote the entry in the i^{th} row, and j^{th} column of A_n by $f_{ij}(x)$ where $f_{ij}(x)$ is defined for all x. Then we have the following theorem:

Theorem 2.1. *Each zero of $F(x)$ satisfies at least one of the n inequalities given by:*

$$|f_{ii}(x)| \leq \sum_{\substack{1 \leq j \leq n \\ j \neq i}} |f_{ij}(x)|, \quad i = 1, 2, ..., n.$$

Proof. We will prove this by contradiction. Assume the opposite, that some root, r, of $F(x)$ satisfies:

$$|f_{ii}(r)| > \sum_{\substack{1 \leq j \leq n \\ j \neq i}} |f_{ij}(r)|, \quad i = 1, 2, ..., n.$$

However if this is the case, then A_n by definition is a strictly diagonally dominant matrix. Since $|A_n| = F(r) = 0$, a contradiction occurs because all strictly diagonally dominant matricies are non-singular [HJ, Theorem 6.1.10].

□

Additionally, since the determinant of a square matrix is equal to the determinant of its transpose we have the following corollary:

Corollary 2.2. *Each zero of $F(x)$ also satisfies at least one of the n equalities given by:*

$$|f_{jj}(x)| \leq \sum_{\substack{1\leq i\leq n \\ i\neq j}} |f_{ij}(x)|, \quad j = 1, 2, ..., n.$$

2.1 Comparison with the Circle Theorem

As shown in [WH], one can use Gerŝchgorin's Circle Theorem to bound the eigenvalues of a scalar, square matrix and therefore bound the roots of the associated characteristic polynomial. In their paper, they find bounds for the zeros of a Fibonacci-type polynomial which exactly agrees with the predictions of Theorem 2.1. However, using Gerŝchgorin's Circle theorem involves unnecassary steps and also has limitations, especially when the problem of finding a corresponding scalar matrix is difficult. Consider the following example:

$$G_n(x) = x^2 G_{n-1}(x) + G_{n-2}(x); \quad G_0(x) = 1, \ G_1(x) = x.$$

It is easily checked by induction that

$$G_n(x) = |B| = \begin{vmatrix} x & -1 & & & \\ 1 & x^2 & -1 & & \\ & 1 & x^2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & 1 & x^2 \end{vmatrix}.$$

If one proceeds with the aim to use Gerŝchgorin's Circle Theorem, they will soon run into difficulties since $G_n(x)$ is the characteristic polynomial of C if

$$G_n(x) = |B| = \det(\lambda I - C).$$

This is problematic because no substitution for λ will make C a scalar matrix, and the theorem cannot be applied. However given $G_n(x) = |B|$, one can directly use Theorem 2.1 to obtain that the zeros of $G_n(x)$ satisfy at least one of the following inequalities:

$$|x| \leq 1; \ |x^2| \leq 2; \ |x^2| \leq 1.$$

Therefore all the roots satisfy

$$|x| \leq \sqrt{2}.$$

3 General Recursions

For a general reccurance given by

$$F_n(x) \;=\; g(x)F_{n-1}(x) \;+\; h(x)F_{n-2}(x),$$

with initial conditions $F_0(x)$ and $F_1(x)$, we can show by induction that $F_n(x)$ is the determinant of the $n \times n$ matrix

$$\begin{vmatrix} F_1(x) & -F_0(x) & & & \\ h(x) & g(x) & -1 & & \\ & h(x) & g(x) & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & h(x) & g(x) \end{vmatrix}.$$

Using this determinant representation of $F_n(x)$ and Theorem 2.1 we can see that all the roots of $F_n(x)$ satisfy at least one of the following inequalities.

$$|F_1(x)| \leq |F_0(x)| \qquad |g(x)| \leq |h(x)| + 1$$

Furthermore, we can use these inequalities to obtain a sufficent condition for boundedness of the roots for $F_n(x)$.

Theorem 3.1. *Let $F_n(x)$ be a recursion with initial conditions $F_0(x)$ and $F_1(x)$, and for all $n \geq 2$*

$$F_n(x) = g(x)F_{n-1}(x) + h(x)F_{n-2}.$$

Then if the values of x that satisfy either $|F_1(x)| \leq |F_0(x)|$ or $|g(x)| \leq |h(x)| + 1$ are bounded in a region in the complex plane, then the roots of $F_n(x)$ are bounded and are located in that region.

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References

[HB] Verner Hoggatt, Jr. and Marjorie Bicknell. Roots of Fibonacci polynomials. *Fibonacci Quart.*, 11(3):271–274, 1973.

[HJ] Roger Horn and Charles R. Johnson. *Matrix analysis*. Cambridge University Press, Cambridge, 1990. Corrected reprint of the 1985 original.

[Mát] Ferenc Mátyás. Bounds for the zeros of Fibonacci-like polynomials. *Acta Acad. Paedagog. Agriensis Sect. Mat. (N.S.)*, 25:15–20 (1999), 1998.

[Moo] Gregory Moore. The limit of the golden numbers is $3/2$. *Fibonacci Quart.*, 32(3):211–217, 1994.

[MZ1] Robert Molina and Aklilu Zeleke. On the convergence of the maximum roots of a Fibonacci-type polynomial sequence. In *Proceedings of the Thirty-Eighth Southeastern International Conference on Combinatorics, Graph Theory and Computing*, volume 184, pages 121–128, 2007.

[MZ2] Robert Molina and Aklilu Zeleke. Generalizing results on the convergence of the maximum roots of Fibonacci type polynomials. In *Proceedings of the Fortieth Southeastern International Conference on Combinatorics, Graph Theory and Computing*, volume 195, pages 95–104, 2009.

[Pro] Helmut Prodinger. The asymptotic behavior of the golden numbers. *Fibonacci Quart.*, 34(3):224–225, 1996.

[Ric] Paolo Ricci. Generalized Lucas polynomials and Fibonacci polynomials. *Riv. Mat. Univ. Parma (5)*, 4:137–146 (1996), 1995.

[WH] Yi Wang and Mingfeng He. Zeros of a class of Fibonacci-type polynomials. *Fibonacci Quart.*, 42(4):341–347, 2004.

[YWH] Hongquan Yu, Yi Wang, and Mingfeng He. On the limit of generalized golden numbers. *Fibonacci Quart.*, 34(4):320–322, 1996.