PROPERTIES OF THE ROOTS OF TRIBONACCI-TYPE POLYNOMIALS

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ABSTRACT. Consider Tribonacci-type polynomials defined by the following recurrence relation

\[ T_n(x) = \alpha(x) \cdot T_{n-1}(x) + \beta(x) \cdot T_{n-2}(x) + \gamma(x) \cdot T_{n-3}(x) \]

where recurrence coefficients \( \alpha(x) \), \( \beta(x) \), and \( \gamma(x) \) and initial conditions \( T_0(x), T_1(x), \) and \( T_2(x) \) are arbitrary functions of \( x \). In this talk, we present matrix representations of \( T_n(x) \), namely \( M_n(x) \), such that \( \det|M_n(x)| = T_{n-1}(x) \). Using this determinant representation, we discuss the nature of all roots of all polynomial sequences of this form using an alternative method of Geršgorin’s Circle Theorem, Laguerre’s application of Samuelson’s Inequality, and an application of Rouché’s Theorem. Special cases of \( T_n(x) \) mentioned include a recurrence with only real roots and a recurrence with only complex roots. We conclude with a presentation of ordinary generating functions for all polynomials mentioned.

1. Introduction

Discovered by Italian mathematician Leonardo Bigollo, the Fibonacci sequence is quite possibly the most famous sequence of numbers. Starting with initial conditions \( F_0 = 0 \) and \( F_1 = 1 \), it follows the recurrence \( F_n = F_{n-1} + F_{n-2} \). Similar to this sequence is the Tribonacci sequence. Beginning with initial values \( T_0 = 1 \), \( T_1 = 1 \), and \( T_2 = 2 \) and recurrence \( T_n = T_{n-1} + T_{n-2} + T_{n-3} \). Aside from being a mathematical curiosity, the Fibonacci numbers show up countless times in nature, two such examples being the arrangement of petals on a flower and the spirals on a pine cone [?]. Similar to the Golden Ratio, which is the ratio of two consecutive Fibonacci numbers and is approximated by 1.618, the Tribonacci Constant is the ratio of two consecutive Tribonacci numbers. It is approximated by 1.839 and plays a prominent role in research about the snub cube [?].

Easily translated into polynomial recurrences, the standard Fibonacci polynomials are defined as having initial conditions of \( F_0(x) = 1 \) and \( F_1(x) = x \) and following the recurrence \( F_n(x) = x \cdot F_{n-1}(x) + F_{n-2}(x) \). In a similar fashion, the standard Tribonacci polynomials have initial conditions \( R_0 = 0, R_1 = 1, R_2 = x^2 \) and following \( R_n(x) = x^2 \cdot R_{n-1}(x) + x \cdot R_{n-2}(x) + R_{n-3}(x) \). Initially published in 1973, Hoggett and Bicknell [?], were able to find an equation for the roots of the Fibonacci polynomials through derivations from hyperbolic trigonometric functions. Gregory A. Moore [?] was able to show in 1992 that for specific initial conditions and recurrence coefficients of Fibonacci-type polynomials that the limit of the maximal real roots is \( \frac{3}{2} \). In 2013, Fricker and Mikus [?] found a new way to bound the roots of Fibonacci-type polynomials. The goal of this paper is to extend their work to Tribonacci-type polynomials.

We consider the standard Tribonacci polynomial recurrence and look at what happens when we change recurrence coefficients and initial conditions. We first
considered only changing the initial conditions, and we were able to find a \( n \times n \) matrix such that the determinant was the \( n - 1 \) polynomial and a bound for the roots using a theorem presented by Fricker and Mikus. We then changed the recurrence coefficients to arbitrary functions of \( x \) and letting the initial conditions be equal to each other and also be an arbitrary function of \( x \). As with the previous conditions, we were able to find a matrix-determinant representation of the polynomial recurrence and a bound for the roots. We were then interested in finding a recurrence with only real roots. We used a theorem from David Kurtz [?] to find sufficient conditions for the recurrence coefficients and initial conditions in order to accomplish this. A matrix-determinant representation, bounds for the roots found by using the theorem from Fricker and Mikus, Laguerre’s interpretation of Samuelson’s inequality [?] as well as a consequence of Rouché’s Theorem [?] are presented. From this, we move to the general case. A matrix-determinant form of the polynomial recurrence as well as a bound for the roots are shown. The paper concludes with a presentation of generating functions for all recurrences mentioned.

2. Some Special Cases

Consider a recursive Tribonacci-type polynomial such that \( P_0(x) = P_1(x) = P_2(x) = 1 \) and \( P_n(x) = x^2 \cdot P_{n-1}(x) + x \cdot P_{n-2}(x) + P_{n-3}(x) \). We then have the following theorem:

**Theorem 1.** The following \( n + 1 \times n + 1 \) matrix is the matrix-determinant form of \( P_n(x) \). That is to say, the determinant of this matrix is \( P_n(x) \). Also note that all blank spaces in all matrices in this paper represent zeros.

\[
\begin{bmatrix}
1 & 1 & 1 \\
x^2 - 1 & x^2 & 1 \\
1 & -x & x^2 & 1 \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
1 & -x & x^2 & 1 \\
1 & -x & x^2 \\
\end{bmatrix}
\]

**Proof.** We begin this proof by displaying the first five polynomials this recurrence generates:

\[
P_0(x) = P_1(x) = P_2(x) = 1 \\
P_3(x) = x^2 + x + 1 \\
P_4(x) = x^2(x^2 + x + 1) + x + 1 = x^4 + x^3 + x^2 + x + 1
\]

We continue this proof with induction. Base cases for \( P_0 \) and \( P_1 \) are trivial, while base cases for \( n = 2, 3, 4 \) are shown below:

\[
det \begin{bmatrix}
1 & 1 & 0 \\
x^2 - 1 & x^2 & 1 \\
0 & x^2 - 1 & x^2 \\
\end{bmatrix} = 1 = P_2(x)
\]
det \begin{bmatrix} 1 & 1 & 0 & 0 \\ x^2 - 1 & x^2 & 1 & 0 \\ 0 & x^2 - 1 & x^2 & 1 \\ 0 & 1 & -x & x^2 \end{bmatrix} = x^2 + x + 1 = P_3(x)

det \begin{bmatrix} 1 & 1 & 0 & 0 \\ x^2 - 1 & x^2 & 1 & 0 \\ 0 & x^2 - 1 & x^2 & 1 \\ 0 & 1 & -x & x^2 \end{bmatrix} = x^4 + x^3 + x^2 + x + 1 = P_4(x)

Now assuming the induction hypothesis, we show that the theorem holds for the \( n + 1 \times n + 1 \) matrix.

Cofactor expansion along the last column gives:

\[
\begin{bmatrix}
1 & x^2 - 1 & x^2 & 1 \\
x^2 - 1 & x^2 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
1 & -x & x^2 & 1
\end{bmatrix}
\begin{bmatrix}
x^2 - 1 & x^2 & 1 \\
x^2 - 1 & x^2 & 1 \\
\cdot & \cdot & \cdot & \cdot \\
1 & -x & x^2 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 0 & 0 \\
x^2 - 1 & x^2 & 1 & 0 \\
\cdot & \cdot & \cdot & \cdot \\
1 & 0 & 1 & -x
\end{bmatrix}
\begin{bmatrix}
x^2 - 1 & x^2 & 1 \\
x^2 - 1 & x^2 & 1 \\
\cdot & \cdot & \cdot & \cdot \\
1 & 0 & 1 & -x
\end{bmatrix}
\]

Notice that the first term is just \( x^2 \cdot P_{n-1}(x) \).

Cofactor expanding the second matrix along the last column gives

\[
\begin{bmatrix}
1 & x^2 - 1 & x^2 & 1 \\
x^2 - 1 & x^2 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
1 & -x & x^2 & 1
\end{bmatrix}
\begin{bmatrix}
x^2 - 1 & x^2 & 1 \\
x^2 - 1 & x^2 & 1 \\
\cdot & \cdot & \cdot & \cdot \\
1 & -x & x^2 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 0 & 0 \\
x^2 - 1 & x^2 & 1 & 0 \\
\cdot & \cdot & \cdot & \cdot \\
1 & 0 & 1 & -x
\end{bmatrix}
\begin{bmatrix}
x^2 - 1 & x^2 & 1 \\
x^2 - 1 & x^2 & 1 \\
\cdot & \cdot & \cdot & \cdot \\
1 & 0 & 1 & -x
\end{bmatrix}
\]

Notice that the last term becomes \( x \cdot P_{n-2}(x) \).

One last cofactor expansion along the bottom row of the first matrix from directly above gives

\[
\begin{bmatrix}
1 & x^2 - 1 & x^2 & 1 \\
x^2 - 1 & x^2 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
1 & -x & x^2 & 1
\end{bmatrix}
\begin{bmatrix}
x^2 - 1 & x^2 & 1 \\
x^2 - 1 & x^2 & 1 \\
\cdot & \cdot & \cdot & \cdot \\
1 & -x & x^2 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 0 & 0 \\
x^2 - 1 & x^2 & 1 & 0 \\
\cdot & \cdot & \cdot & \cdot \\
1 & 0 & 1 & -x
\end{bmatrix}
\begin{bmatrix}
x^2 - 1 & x^2 & 1 \\
x^2 - 1 & x^2 & 1 \\
\cdot & \cdot & \cdot & \cdot \\
1 & 0 & 1 & -x
\end{bmatrix}
\]

Combining the results gives

\[ P_n(x) = x^2 \cdot P_{n-1}(x) + x \cdot P_{n-2}(x) + P_{n-3}(x). \]
A major topic of this paper is to remark upon the roots of Tribonacci-type polynomials. To accomplish this, we use the following theorem to establish the positivity of $P_n(x)$.

**Theorem 2.** $P_n(x)$ is positive for all values of $x$.

**Proof.** It is sufficient to show $|x \cdot P_{n-2}(x)| < x^2 \cdot P_{n-1}(x) + P_{n-3}(x)$. This is proved by inducting upon $n$.

Base Case: $|x \cdot P_2(x)| = |x| < x^2 + 1 = x^2 \cdot P_1(x) + P_3(x)$

Induction Step:

$$|x \cdot P_{n-2}(x)| = |x(x^2 \cdot P_{n-3}(x) + x \cdot P_{n-4}(x) + P_{n-5}(x))|$$

$$\leq |x^3 \cdot P_{n-3}(x)| + |x^2 \cdot P_{n-4}(x)| + |x \cdot P_{n-5}(x)|$$

$$< x^2(x^2 \cdot P_{n-3}(x) + P_{n-4}(x)) + x(x^2 \cdot P_{n-3}(x) + P_{n-5}(x)) + x^2 \cdot P_{n-4}(x) + P_{n-6}(x)$$

$$= x^4 \cdot P_{n-2}(x) + x^2 \cdot P_{n-4}(x) + x^3 \cdot P_{n-3}(x) + x \cdot P_{n-5}(x) + x^2 \cdot P_{n-4}(x) + P_{n-6}(x)$$

$$= x^4 \cdot P_{n-2}(x) + x^3 \cdot P_{n-3}(x) + x^2 \cdot P_{n-3}(x) + P_{n-3}(x)$$

$$= x^2(x^2 \cdot P_{n-2}(x) + x \cdot P_{n-3}(x) + P_{n-4}(x)) + P_{n-3}(x)$$

$$= x^2 \cdot P_{n-1}(x) + P_{n-3}(x)$$

\[\square\]

**Corollary 1.** All roots of $P_n(x)$ are complex.

**Proof.** This comes as a result of $P_n(x)$ being positive everywhere. \[\square\]

As mentioned before, facts about the roots of Tribonacci-type polynomials are of importance. A theorem, call it the Root Bounding Theorem, found by [FM] Fricker and Mikus states that for some polynomial, $F(x)$, that can be represented by the determinant of a $n \times n$ matrix, $M_n$, with $f_{ij}$ the $i^{th}$-row and $j^{th}$-column entry of $M_n$, the zeros of $F(x)$ satisfy at least one of the $n$ inequalities given by:

$$|f_{ii}(x)| \leq \sum_{1 \leq j \leq n} |f_{ij}(x)|, \forall i = 1, 2, ..., n$$

**Theorem 3.** The roots of $P_n(x)$ are bounded by 2.

**Proof.** From the determinant matrix and equation (1), we get the following inequalities:

1. $1 \leq 1$
2. $|x^2| \leq x^2 + 2$
3. $|x^2| \leq |x| + 2$
4. $|x^2| \leq |x| + 1$

Simplification of those inequalities provides $x \leq 1$ or $x \leq 2$. \[\square\]

If we change the initial conditions of $P_n(x)$ to $P_0(x) = P_1(x) = P_2(x) = a$, for $a \in \mathbb{C}$ and $a \neq 0$, but keep the coefficients of recurrence the same, it can be shown
that the roots of this new polynomial recurrence, call it $A_n(x)$, are also bounded by 2.

**Theorem 4.** The roots of $A_n(x)$ are bounded by 2.

**Proof.** The key to this theorem is noticing that $A_n(x) = a \cdot P_n(x)$. Once that is shown, it easily follows that the roots of $A_n(x)$ are bounded by 2.

The proof for base cases $A_0(x) - A_2(x)$ are trivial but are shown below:

$$A_0(x) = A_1(x) = A_2(x) = a = a \cdot 1 = a \cdot P_0(x) = a \cdot P_1(x) = a \cdot P_2(x)$$

We continue the proof by using induction.

$$A_n(x) = x^2 \cdot A_{n-1}(x) + x \cdot A_{n-2}(x) + A_{n-3}(x)$$
$$= x^2(a \cdot P_{n-1}(x)) + x(a \cdot P_{n-2}(x)) + a \cdot P_{n-3}(x)$$
$$= a[x^2 \cdot P_{n-1}(x) + x \cdot P_{n-2}(x) + P_{n-3}(x)]$$
$$= a \cdot P_n(x)$$

Since $A_n(x) = a \cdot P_n(x)$, when we try to find the roots of each $A_n(x)$, we get

$$0 = a \cdot P_n(x)$$

Dividing both sides by $a$ gives

$$0 = P_n(x)$$

This equality provides the roots for $P_n(x)$, which were shown to be bounded by 2. □

### 3. Special Case 2

We continue on to our goal of getting to the recurrence with arbitrary coefficients and initial conditions by presenting results about a recurrence with mostly arbitrary coefficients and conditions. Consider the recurrence relation $O_n(x) = \alpha(x) \cdot O_{n-1}(x) + \beta(x) \cdot O_{n-2}(x) + \gamma(x) \cdot O_{n-3}(x)$ with initial conditions $O_0(x) = O_1(x) = O_2(x) = l(x)$, where $\alpha(x)$, $\beta(x)$, $\gamma(x)$, and $l(x)$ are arbitrary functions of $x$. Presented below are the matrix-determinant form of the polynomial recursion and a bound for the roots.

**Theorem 5.** The below matrix is the matrix-determinant representation of $O_n(x)$.

$$\begin{bmatrix}
 l(x) & l(x) \\
 \alpha(x) - 1 & \alpha(x) & 1 \\
 \alpha(x) - 1 & \alpha(x) & 1 \\
 \gamma(x) & -\beta(x) & \alpha(x) & 1 \\
 \vdots & \ddots & \ddots & \ddots & \ddots \\
 \gamma(x) & -\beta(x) & \alpha(x) & 1 \\
 \gamma(x) & -\beta(x) & \alpha(x) & 1
\end{bmatrix}$$

**Proof.** Induction and co-factor expansion are the keys to this proof, which similarly follows the proof for Theorem 1. The full proof for this theorem can be found in the Appendix. □
Corollary 2. The roots of $O_n(x)$ are bound by at least one of the following inequalities:

1. $|l(x)| \leq |l(x)|$
2. $|\alpha(x)| \leq |\alpha(x)| + 2$
3. $|\alpha(x)| \leq |\gamma(x)| + |\beta(x)| + 1$
4. $|\alpha(x)| \leq |\gamma(x)| + |\beta(x)|$

Proof. The inequalities were found using the Root Bounding Theorem. \hfill \Box

4. Special Case 3

Consider $V_n(x) = x^2 \cdot \sum_{i=0}^{k}(a + i) + x \cdot \sum_{i=0}^{k}(a + c + 2i) + \sum_{i=0}^{k}(c + i)$ with initial conditions $V_0 = a$, $V_1 = b$, and $V_2 = c$. The goal of this section is to provide a bound for the roots of $V_n(x)$. To do this, we first show that all roots of $V_n(x)$ are real. We use the following theorem from Kurtz [7] to accomplish this.

It states that a polynomial, $K_n(x)$, of degree $n \geq 2$ with positive coefficients where

$$a_i^2 - 4 \cdot a_{i-1} \cdot a_{i+1} > 0, \quad i = 1, 2, ..., n - 1$$

has all real and distinct roots.

Theorem 6. All roots of $V_n(x)$ are real and distinct.

Proof. Here, it suffices to show that

$$\left( \sum_{i=0}^{k}(a + c + 2i) \right)^2 - 4 \cdot \sum_{i=0}^{k}(a + i) \cdot \sum_{i=0}^{k}(c + i) > 0.$$ 

Applying the summation identities, we can change the above equation into:

$$(na + nc + n(n + 1))^2 - 4 \cdot (na + \frac{n(n + 1)}{2}) \cdot (nc + \frac{n(n + 1)}{2})$$

Carrying out the multiplication and summation, while also simplifying gives:

$$a^2 n^2 + c^2 n^2 - 2acn^2 = n^2(a - c)^2$$

Since $n > 0$ and $a \neq c$, we know that $n^2(a - c)^2 > 0$. \hfill \Box

Because $V_n(x)$ has only real roots, we can use Laguerre’s application of Samuelson’s Inequality and a consequence of Rouché’s Theorem to bound the roots.

Laguerre says that if

$$\sum_{k=0}^{n} a_k \cdot x^k$$

is a polynomial with only real roots, then all of its roots are in the interval

$$x_{\pm} = \frac{a_{n-1}}{a_n} \pm \frac{n - 1}{n \cdot a_n} \sqrt{a_{n-1}^2 - \frac{2n}{n - 1} \cdot a_n \cdot a_{n-2}}$$

Theorem 7. Applying Laguerre’s equation to $V_n(x)$ gives that as $n, k \to \infty$, the roots of $V_{n,k}(x)$ converge to $-1$. 

Proof.

\[
\lim_{n,k \to \infty} x_{\pm} = \frac{T_n \sum_{i=0}^{k} a + c + 2i}{2T_n \sum_{i=0}^{k} a + i} \pm \frac{1}{2T_n \sum_{i=0}^{k} a + i} \sqrt{T_n \sum_{i=0}^{k} a + c + 2i - 4(T_n \sum_{i=0}^{k} a + i)(T_n \sum_{i=0}^{k} c + i)}
\]

Roché’s Theorem states that for complex-valued functions \( f(x) \) and \( g(x) \), that are holomorphic on and inside some closed contour \( K \), where \( |g(x)| < |f(x)| \), then \( f(x) \) and \( f(x) + g(x) \) have the same number of roots within \( K \).

Letting

\[
f(x) = a_j R^j(x)
\]

\[
g(x) = a_0 + a_1 R(x) + \ldots + a_{j-1} R^{j-1}(x) + a_j R^j(x) + \ldots + a_n R^n(x),
\]

we get that

\[
f(x) + g(x) = \sum_{i=0}^{n} a_i R^i(x).
\]

By Roché’s Theorem, we can then say that \( f(x) + g(x) \) have the same number of roots as \( f(x) \), and then by simple observation we see that a root to \( f(x) + g(x) \) is going to be

\[
1 + \frac{\max\{|a_0|, |a_1|, \ldots, |a_{n-1}|\}}{|a_n|}
\]

**Theorem 8.** An application of Roché’s Theorem gives that as \( n, k \to \infty \), the roots of \( V_{n,k}(x) \) are contained in the interval \([-3,3]\).
Proof.

\[
\lim_{n,k \to \infty} \rho_2 = 1 + \max_{0 \leq i \leq n-1} \left( \frac{T_n \sum_{i=0}^{k} a + c + 2i}{T_n \sum_{i=0}^{k} a + i} \right) \]

\[
= 1 + \frac{T_n \sum_{i=0}^{k} a + c + 2i}{T_n \sum_{i=0}^{k} a + i} = 1 + T_n \frac{ka + kc + k(k+1)}{ka + \frac{k(k+1)}{2}} = 1 + \frac{ka + kc + k(k+1)}{2} \sum_{i=0}^{k} \frac{1}{2}
\]

\[
= 1 + 2 \frac{ka + kc + k(k+1)}{2} = 1 + 2 = 3
\]

□

5. GENERAL CASE

Now presented are the results for the recurrence \( T_n(x) = \alpha(x) \cdot T_{n-1}(x) + \beta(x) \cdot T_{n-2}(x) + \gamma(x) \cdot T_{n-3}(x) \) with \( T_0(x) = \alpha(x), T_1(x) = b(x), \) and \( T_2(x) = c(x) \), where \( \alpha(x), \beta(x), \gamma(x), a(x), b(x), \) and \( c(x) \) are arbitrary functions of \( x \).

**Theorem 10.** The below matrix is the matrix-determinant form of \( T_n(x) \).

\[
\begin{pmatrix}
1 & \alpha(x) - b(x) & 1 \\
\alpha(x) & b(x) \cdot \alpha(x) - c(x) & \alpha(x) \\
\alpha(x) \cdot \gamma(x) & -\beta(x) & \alpha(x) \\
\gamma(x) & -\beta(x) & \alpha(x) \\
\gamma(x) & -\beta(x) & \alpha(x) \\
\gamma(x) & -\beta(x) & \alpha(x)
\end{pmatrix}
\]

**Proof.** The major difference between this matrix and the previous three matrices is that this matrix only holds for the 2 x 2 case and above. Otherwise, the proof of this theorem follows the same steps as the proof for Theorem 1. The full proof for this theorem can be found in the appendix. □

**Corollary 5.** The roots of \( T_n(x) \) are bounded by the following inequalities:

1. \( 1 \leq 1 \)
2. \( |\alpha(x)| \leq |\alpha(x)| + | - b(x)| + 1 \)
3. \( |\alpha(x)| \leq |b(x) \cdot \alpha(x) - c(x)| + 1 \)
Proof. The inequalities come from the Root Bounding Theorem.

6. Ordinary Generating Functions

Generating Functions have many applications in various fields of study including combinatorics and physics. They can be used to solve linear recurrence problems, as well as to find the number of partitions of a natural number. They also are intimately related to electromagnetism.[?] They often come in two forms: A summation notation and a closed form of a single variable. The goal here is to find the closed form of the generating function, $F(s) := \sum_{i=0}^{n} s^i \cdot T_i(x)$, and to modify it for other Tribonacci recurrences.

**Theorem 11.** The generating function for $T_n(x)$ is as follows:

$$F(s) = \frac{a + b \cdot s + c \cdot s^2 - a \cdot \alpha(x) \cdot s - b \cdot \alpha(x) \cdot s^2 - a \cdot \beta(x) \cdot s^2}{1 - \alpha(x) \cdot s - \beta(x) \cdot s^2 - \gamma(x) \cdot s^3}$$

**Proof.** We begin this proof by writing $T_{n+3}(x)$ as

$$T_{n+3}(x) = \alpha(x) \cdot T_{n+2} + \beta(x) \cdot T_{n+1} + \gamma(x) \cdot T_{n}(x)$$

Following this, we multiply both sides by $s^{n+3}$ to get

$$T_{n+3}(x) \cdot s^{n+3} = \alpha(x) \cdot T_{n+2} \cdot s^{n+3} + \beta(x) \cdot T_{n+1} \cdot s^{n+3} + \gamma(x) \cdot T_{n}(x) \cdot s^{n+3}$$

We now take a summation of both sides to get

$$\sum_{n=0}^{\infty} T_{n+3}(x) \cdot s^{n+3} = \sum_{n=0}^{\infty} \alpha(x) \cdot T_{n+2} \cdot s^{n+3} + \sum_{n=0}^{\infty} \beta(x) \cdot T_{n+1} \cdot s^{n+3} + \sum_{n=0}^{\infty} \gamma(x) \cdot T_{n}(x) \cdot s^{n+3}$$

$$= s \cdot \alpha(x) \sum_{n=0}^{\infty} T_{n+2} \cdot s^{n+2} + s^2 \cdot \beta(x) \sum_{n=0}^{\infty} T_{n+1} \cdot s^{n+1} + s^3 \cdot \gamma(x) \sum_{n=0}^{\infty} T_{n}(x) \cdot s^n$$

We now define $F(s)$ to be as follows

$$F(s) := \sum_{n=0}^{\infty} s^n \cdot T_n(x)$$

Notice

$$\sum_{n=0}^{\infty} s^{n+1} T_{n+1}(x) = F(s) - T_0(x)$$

$$\sum_{n=0}^{\infty} s^{n+2} T_{n+2}(x) = F(s) - T_0(x) - s \cdot T_1(x)$$

$$\sum_{n=0}^{\infty} s^{n+3} T_{n+3}(x) = F(s) - T_0(x) - s \cdot T_1(x) - s^2 \cdot T_2(x)$$

Substituting in the noticed values into the summation equation provides:

$$F(s) - T_0(x) - s \cdot T_1(x) - s^2 \cdot T_2(x) = s \cdot \alpha(x) [F(s) - T_0(x) - s \cdot T_1(x)] + s^2 \cdot \beta(x) [F(s) - T_0(x)] + s^3 \cdot \gamma(x) [F(s)]$$
Substituting in the values for $T_0(x)$, $T_1(x)$, and $T_2(x)$ gives

$$F(s) - a - b \cdot s - c \cdot s^2 = s \cdot \alpha(x) [F(s) - a - bs] + s^2 \cdot \beta(x) [F(s) - a] + s^3 \cdot \gamma(x) \cdot F(s)$$

$$= s \cdot \alpha(x) \cdot F(s) - s \cdot a \cdot \alpha(x) - s^2 \cdot b \cdot \alpha(x) + s^2 \cdot \beta(x) \cdot F(s)$$

$$- s^2 \cdot a \cdot \beta(x) + s^3 \cdot \gamma(x) \cdot F(s)$$

Collecting all the terms with $F(s)$ on one side gives:

$$a + bs + cs^2 - s \alpha(x) - s^2 b \alpha(x) - s^2 \alpha(x) = F(s) - s \cdot \alpha(x) \cdot F(s) - s^2 \cdot \beta(x) \cdot F(s) - s^3 \cdot \gamma(x) \cdot F(s)$$

$$= F(s)[1 - s \cdot \alpha(x) - s^2 \cdot \beta(x) - s^3 \cdot \gamma(x)]$$

Solving for $F(s)$ then provides:

$$F(s) = \frac{a + b \cdot s + c \cdot s^2 - s \cdot a \cdot \alpha(x) - s^2 \cdot b \cdot \alpha(x) - s^2 \cdot a \cdot \beta(x)}{1 - s \cdot \alpha(x) - s^2 \cdot \beta(x) - s^3 \cdot \gamma(x)}$$

□

**Corollary 6.** The generating functions for $P_n(x)$, $A_n(x)$, and $O_n(x)$ are shown below in their respective order:

$$P(s) = \frac{(1 - x - x^2)s^2 + (1 - x^2)s + 1}{1 - s \cdot x^2 - s^2 \cdot x - s^3}$$

$$A(s) = \frac{(a - ax^2 - ax)x^2 + (a - ax^2)s + a}{1 - s \cdot x^2 - s^2 \cdot x - s^3} = a \cdot P(s)$$

$$O(s) = \frac{(l(x) - l(x)\alpha(x) - l(x)\beta(x))s^2 + (l(x) - l(x)\alpha(x))s + l(x)}{1 - \alpha(x) \cdot s - \beta(x) \cdot s^2 - \gamma(x) \cdot s^3}$$

**Proof.** The above generating functions were found as a result of plugging in values for $\alpha(x), \beta(x), \gamma(x), f(x), g(x)$, and $h(x)$ in the generating function for the $T_n(x)$.

□

An interesting thing to note about the generating functions is that the roots of the polynomial in front of the $s^2$ term in the numerator of $P(s)$ is the golden ratio.

7. **Acknowledgements**

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**References**
8. Appendix

Theorem 5. The below \( n = 1 \times n + 1 \) matrix is the matrix-determinant representation of \( O_n(x) \), for \( O_n(x) = \alpha(x) \cdot O_{n-1}(x) + \beta(x) \cdot O_{n-2}(x) + \gamma(x) \cdot O_{n-3}(x) \) with initial conditions \( O_0(x) = O_1(x) = O_2(x) = l(x) \), where \( \alpha(x), \beta(x), \gamma(x) \), and \( l(x) \) are arbitrary functions of \( x \):

\[
\begin{bmatrix}
  l(x) & l(x) \\
  \alpha(x) & \alpha(x) - 1 & 1 \\
  \gamma(x) & -\beta(x) & \alpha(x) & 1 \\
  \vdots & \ddots & \ddots & \ddots & \ddots \\
  \gamma(x) & -\beta(x) & \alpha(x) & 1 \\
  \gamma(x) & -\beta(x) & \alpha(x) & 1 \\
\end{bmatrix}
\]

Proof. Base cases for \( O_0(x) \) and \( O_1(x) \) are trivial, while base case examples for \( O_2(x) \) and \( O_3(x) \) are shown below.

\[
det \begin{bmatrix}
  l(x) & l(x) \\
  \alpha(x) & \alpha(x) - 1 & 1 \\
  \gamma(x) & -\beta(x) & \alpha(x) & 1 \\
\end{bmatrix} = l(x) = O_2(x)
\]

\[
det \begin{bmatrix}
  a & a \\
  \alpha(x) - 1 & \alpha(x) \\
  \gamma(x) & -\beta(x) & \alpha(x) & 1 \\
\end{bmatrix} = a[\alpha(x) + \beta(x) + \gamma(x)] = O_3(x)
\]

We continue with induction by assuming that the matrix holds for \( n \) and then by taking the determinant of the \( n + 1 \times n + 1 \) matrix.

Cofactor expansion along the last column gives:

\[
\begin{bmatrix}
  l(x) & l(x) \\
  \alpha(x) & \alpha(x) - 1 & 1 \\
  \gamma(x) & -\beta(x) & \alpha(x) & 1 \\
\end{bmatrix} - 1 \cdot det \begin{bmatrix}
  l(x) & l(x) \\
  \alpha(x) & \alpha(x) - 1 & 1 \\
  \gamma(x) & -\beta(x) & \alpha(x) & 1 \\
\end{bmatrix}
\]

Notice that the first term is just \( \alpha(x) \cdot O_{n-1}(x) \).

Cofactor expanding the second column along the last column gives

\[
\begin{bmatrix}
  l(x) & l(x) \\
  \alpha(x) & \alpha(x) - 1 & 1 \\
  \gamma(x) & -\beta(x) & \alpha(x) & 1 \\
\end{bmatrix} + 1 \cdot det \begin{bmatrix}
  l(x) & l(x) \\
  \alpha(x) & \alpha(x) - 1 & 1 \\
  \gamma(x) & -\beta(x) & \alpha(x) & 1 \\
\end{bmatrix}
\]

Notice that the first term is simply \( \beta(x) \cdot O_{n-2}(x) \).

We finish the proof by cofactor expanding the second matrix from above to get
Notice that the first term is just $\gamma$. 

**Cofactor expanding the second matrix along the last column gives** 

$$
\begin{bmatrix}
1 & l(x) & l(x) \\
\alpha(x) & \alpha(x) - 1 & 1 \\
\alpha(x) & \alpha(x) - 1 & 1 \\
\gamma(x) & -\beta(x) & \alpha(x) \\
\gamma(x) & -\beta(x) & \alpha(x) \\
\end{bmatrix}
= \gamma \cdot O_{n-3}(x)
$$

Combining the results gives $O_n(x) = \alpha(x) \cdot O_{n-1}(x) + \beta(x) \cdot O_{n-2}(x) + \gamma(x) \cdot O_{n-3}(x)$.

**Theorem 10.** The below $n + 1 \times n + 1$ matrix is the matrix-determinant form of $T_n(x)$, for $T_n(x) = \alpha(x) \cdot T_{n-1}(x) + \beta(x) \cdot T_{n-2}(x) + \gamma(x) \cdot T_{n-3}(x)$ with $T_0(x) = a(x)$, $T_1(x) = b(x)$, and $T_2(x) = c(x)$, where $\alpha(x), \beta(x), \gamma(x), a(x), b(x)$, and $c(x)$ are arbitrary functions of $x$:

$$
\begin{bmatrix}
1 & a(x) - b(x) & 1 \\
\alpha(x) & b(x) \cdot \alpha(x) - c(x) & \alpha(x) \\
\alpha(x) & a(x) \cdot \gamma(x) - \beta(x) & \alpha(x) \\
\gamma(x) & -\beta(x) \cdot \alpha(x) & 1 \\
\gamma(x) & -\beta(x) \cdot \alpha(x) & 1 \\
\end{bmatrix}
$$

**Proof.** This matrix does not hold for $T_0$, and the proof for $T_1(x)$ is trivial. Base case examples for $T_2(x)$ and $T_3(x)$ are shown below.

$$
\det \begin{bmatrix} 1 & 1 \\ \alpha(x) - b(x) & \alpha(x) \\ b(x) \cdot \alpha(x) - c(x) & \alpha(x) \end{bmatrix} = 1 = T_2(x)
$$

$$
\det \begin{bmatrix} 1 & 1 \\ \alpha(x) - b(x) & \alpha(x) \\ b(x) \cdot \alpha(x) - c(x) & \alpha(x) \end{bmatrix} = c(x) \cdot \alpha(x) + b(x) \cdot \beta(x) + a(x) \cdot \gamma(x) = T_3(x)
$$

We continue with induction by assuming that the matrix holds for $n$ and then by taking the determinant of the $n + 1 \times n + 1$ matrix.

Cofactor expansion along the last column gives:

$$
\begin{bmatrix}
\alpha(x) \\
\alpha(x) \\
\gamma(x) \\
\gamma(x) \\
\end{bmatrix}
\det \begin{bmatrix} 1 & 1 & 1 \\ \alpha(x) - b(x) & \alpha(x) & 1 \\ b(x) \cdot \alpha(x) - c(x) & \alpha(x) & 1 \\ \alpha(x) \cdot \gamma(x) - \beta(x) & \alpha(x) & 1 \\
\gamma(x) - \beta(x) & \alpha(x) & 1 \\
\gamma(x) - \beta(x) & \alpha(x) & 1 \\
\gamma(x) - \beta(x) & \alpha(x) & 1 \\
\gamma(x) - \beta(x) & \alpha(x) & 1 \\
\end{bmatrix}
$$

Notice that the first term is just $\alpha(x) \cdot T_{n-1}(x)$.

Cofactor expanding the second matrix along the last column gives...
Notice that the first term is simply $\beta(x) \cdot T_{n-2}(x)$.
We finish the proof by cofactor expanding the second matrix from above to get

$$
\gamma(x) \cdot \det \begin{pmatrix}
1 & \alpha(x) - b(x) & \alpha(x) & 1 \\
\alpha(x) & b(x) \cdot \alpha(x) - c(x) & \alpha(x) & 1 \\
a(x) \cdot \gamma(x) & a(x) \cdot \gamma(x) - \beta(x) & a(x) & 1 \\
& \ddots & \ddots & \ddots \\
& & & & \\
& & & & \gamma(x) - \beta(x) & \alpha(x) & 1 \\
& & & & \gamma(x) & -\beta(x) & \alpha(x)
\end{pmatrix}
= \gamma(x) \cdot R_{n-3}(x)
$$

Combining the results gives $T_n(x) = \alpha(x) \cdot T_{n-1}(x) + \beta(x) \cdot T_{n-2}(x) + \gamma(x) \cdot T_{n-3}(x)$. \qed

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