# A Particle in Cell Corrected Approach to Direct Sum

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# The N-Body Problem



- Solar systems

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- Interacting charges, gases and plasma

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- Solar systems
- Interacting charges, gases and plasma
- We want to solve the system of ODE's given by:

$$\frac{dx_i}{dt} = v_i \qquad \frac{dv_i}{dt} = a_i = \frac{F_i(x_1, \ldots, x_n)}{m_i}$$

for $i = 1, \ldots, n$.

# Traditional Solutions



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- Direct Sum
- Particle Mesh
- Particle-Particle Particle-Mesh

- The net force on particle $i$ is:

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- To calculate the force on all of the particles, we must repeat this summation for each of the $n$ particles
- The problem with direct sum is that it is $\mathcal{O}(N^2)$

# Goal

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- Find efficient ways to solve the n-body problem.
- Improve efficiency from $\mathcal{O}(n^2)$ to $\mathcal{O}(n) + \mathcal{O}(k^2)$ where $k << n$ and $k$ is the number of particles per cell.

$$E_{j,i} = \frac{q_j}{\epsilon_0}\mathcal{G}(x_j|x_i)$$

$$F_{j,i} = q_i * E_{j,i}$$

Table: Theoretical Time Comparison (3600 MHz computer)

| $N$ | $\mathcal{O}(N^2)$ | $\mathcal{O}(N)$ |
|---|---|---|
| 10 | $10^{-8}$ sec | $10^{-9}$ sec |
| $10^3$ | $10^{-4}$ sec | $10^{-7}$ sec |
| $10^6$ | 5 min | $10^{-4}$ sec |
| $10^9$ | 8 yrs | .28 sec |
| $10^{12}$ | $10^6$ yrs | 4.6 min |

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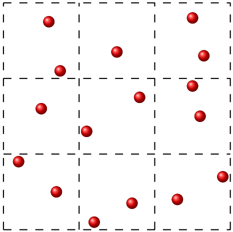

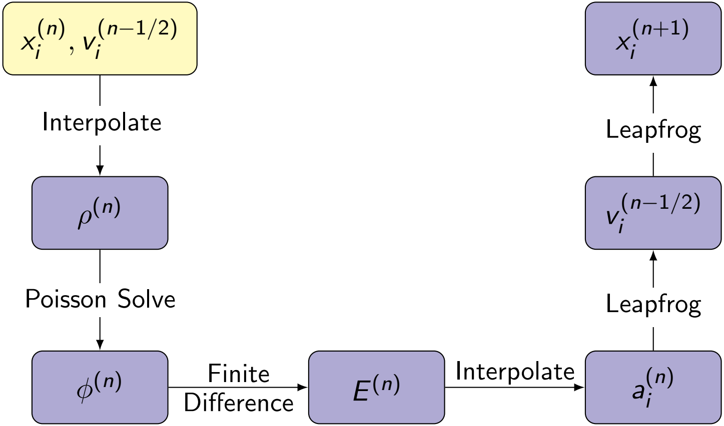
$x_i^{(n)}, v_i^{(n-1/2)}$
Interpolate
$\rho^{(n)}$
Poisson Solve
$\phi^{(n)}$
Finite Difference
$E^{(n)}$
Interpolate
$a_i^{(n)}$
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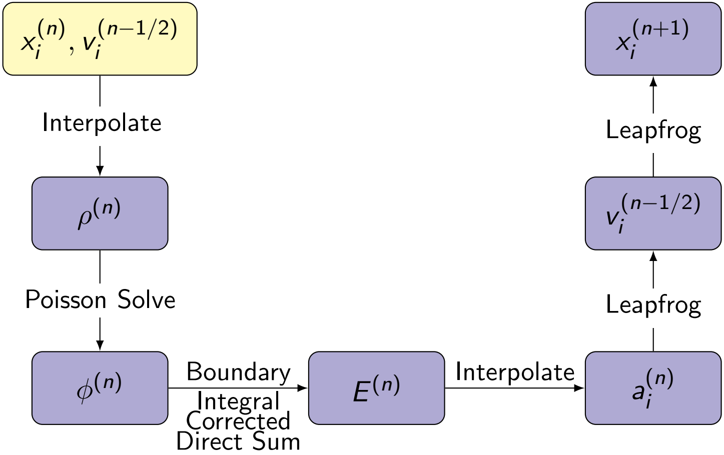
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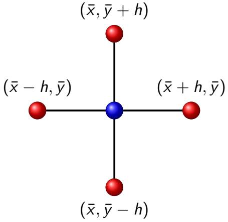
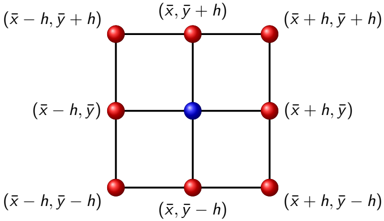
# Using Finite Difference to Solve






**Five-Point Stencil**

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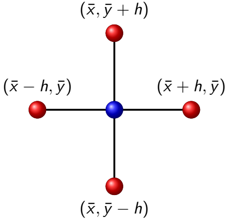


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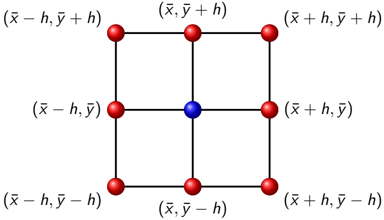


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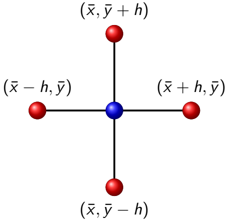


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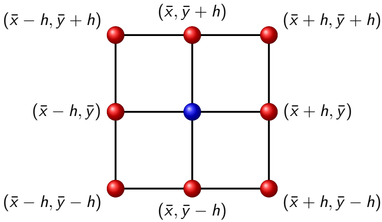


**Nine-Point Stencil**

- $\nabla^2\phi = \partial_{xx}\phi + \partial_{yy}\phi$
- $\nabla^2\phi = \frac{\phi^{j+1,k} - 2\phi^{j,k} + \phi^{j-1,k}}{(\Delta x)^2} + \frac{\phi^{j,k+1} - 2\phi^{j,k} + \phi^{j,k-1}}{(\Delta y)^2} + \mathcal{O}(\Delta x^2 + \Delta y^2)$
  where $j$ is the index for $x$ and $k$ is the index for $y$

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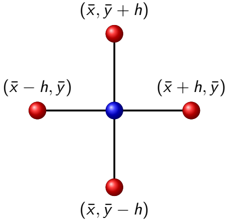


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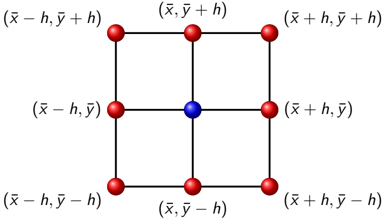


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- $A\vec{\phi} = \vec{\rho}$

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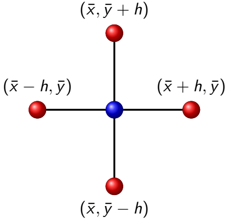


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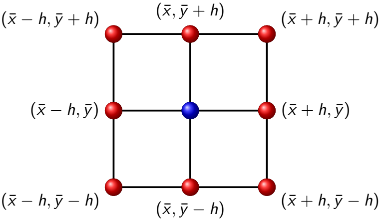


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  where $j$ is the index for $x$ and $k$ is the index for $y$
- $A\vec{\phi} = \vec{\rho}$
- $E_y(x, y) = \frac{\phi(x,y+h) - \phi(x,y-h)}{2h}$,
  $E_x(x, y) = \frac{\phi(x+h,y) - \phi(x-h,y)}{2h}$

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- $v_{1/2} = v_0 + a_0 \cdot \frac{dt}{2}$
- $v_{n+1/2} = v_{n-1/2} + a_n \cdot dt$
- $x_n = x_{n-1} + v_{n-1/2} \cdot dt$

## Deriving Green's Identity

$$\phi(x) = \sum_{i=1}^{N} G(x_i - x_0) - \int_{\partial V} \left(G(x - x_0)\nabla\phi - \phi\nabla G(x - x_0)\right)\cdot\vec{n}ds$$

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Let $\vec{F} = \theta\gamma$ where $\theta$ is a scalar function, $\gamma = \nabla\psi$ is a vector function.

$$\int_V \left(\theta\nabla^2\psi + \nabla\theta \cdot \nabla\psi\right) dV = \int_{\partial V} \left(\theta\nabla\psi \cdot \vec{n}\right) ds$$

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If we let $\theta = G(x - x_0)$ and $\psi = \phi$, then we can solve for the potential $\phi(x)$

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- $\phi_{x,j} = \sum_{\substack{i=1 \\ i\neq j}}^{N} \frac{-1}{2\pi} \ln\left(||x_i - x_j||_2\right) + \oint_{\partial\Omega} (\phi\nabla G - G\nabla\phi) \cdot \vec{n}\, ds$

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- In order to evaluate the boundary integral, we solve using numerical quadrature.
- We end up with a matrix that we have to invert to obtain the surface charge.
- Then, instead of doing this everywhere which would be $\mathcal{O}(n^2)$, we make this a subcell method which gives $\mathcal{O}(n) + \mathcal{O}(k^2)$.

# Solving for Sigma



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- We use integral mean value theorem and trapezoid rule to end up with the Green's matrix
- $\sum_{i=1}^{4k} \int_{s_1}^{s_2} \sigma G \, ds$
- Then we end up with a system of equations in which we need to perform a Green's matrix inversion

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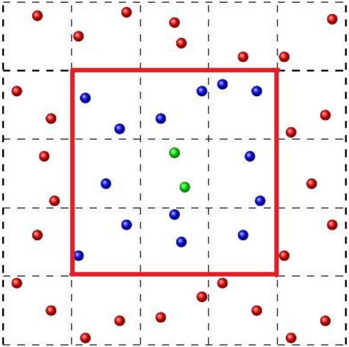
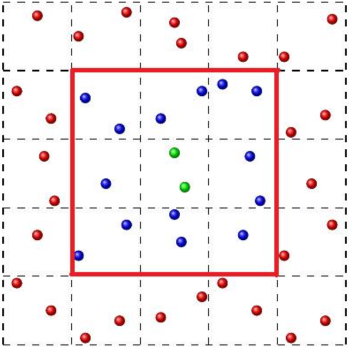
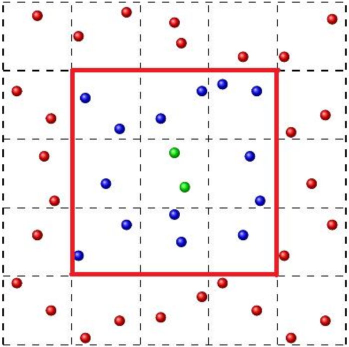
- List of all particles inside the cell



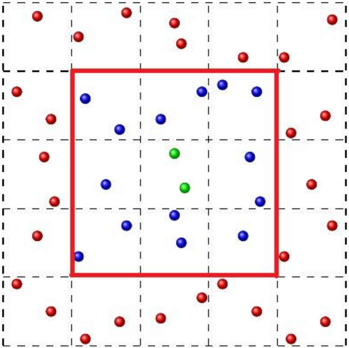
- List of all particles in boundary cells



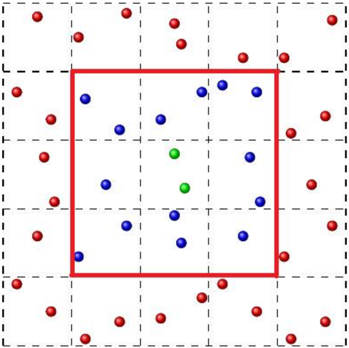
- Potential at all 12 border points per cell



- Something to take the above and calculate the electric field



- For loop over all cells



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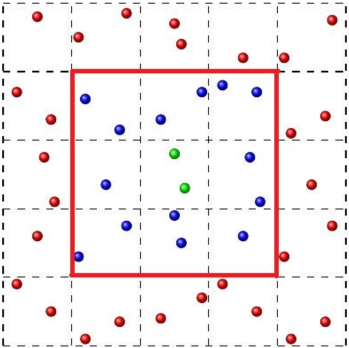

If we can get this done, then it's a matter of increasing customizability

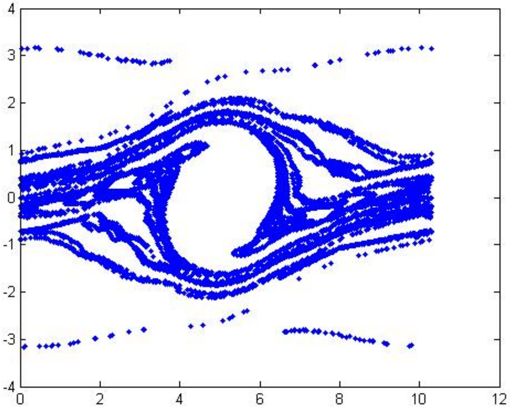
4
3
2
1
0
-1
-2
-3
-4
0
2
4
6
8
10
12

Figure: Uncorrected

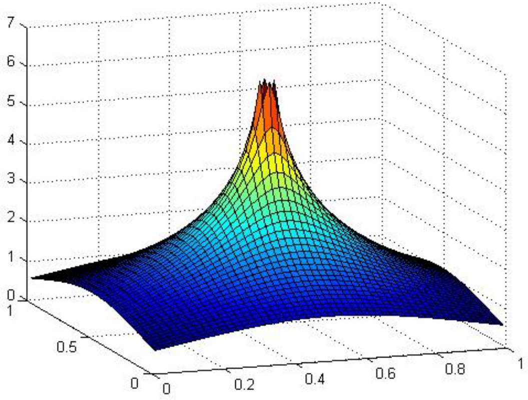

Figure: Point charge at the center