Random Walks on Spheres and Harmonic Functions:

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Harmonic functions arise in heat propagation, diffusion and migration models. We may be interested in temperature (or chemical concentration, or population density) at a particular point of a certain region, given specific conditions at the edge of the region, once equilibrium is reached. This is the Dirichlet problem: for a given region D, solve $\Delta u = 0$ with given boundary condition $f(x)$, where $u(x)$ is, say, the temperature. Numerical methods that provide an answer to this question usually involve constructing a mesh and iteratively solving a system of linear equations.

However, Kakutani [2] proved that a solution can be obtained by considering Brownian motion starting at an interior point of the given region D. The points of “first encounter” of the Wiener process with the boundary induce a distribution on the boundary of D. The expectation of $f(x)$ with respect to this distribution is $u(x_0)$.

Brownian motion describes the random movement of a particle in a fluid or gas. It was first described by Scottish botanist Robert Brown, and is usually modeled and studied using stochastic calculus. The method of Random Walks On Spheres (RWOS), however, efficiently simulates Brownian motion. It is known that if a particle starts at the center of a disk, the points of first encounter of a Brownian motion induce a uniform distribution on the boundary of the disk. To find the distribution on the boundary of a more general region D, we can iteratively choose disks interior to D, choose a point uniformly from the boundary of this disk, and repeat until the particle gets sufficiently close to the boundary of D.

Using the above method we estimate the temperature (or concentration, or population density) at any point x. The Monte Carlo algorithm to be implemented by participants in a number of group projects relies on three major properties of Brownian motion:

- Under suitable conditions on the region, a particle in Brownian motion reaches the region’s boundary in finite time with probability 1.
- If a particle starts at the center of a disc, the point of exit (where it reaches the circle) is distributed uniformly.
- The boundary value of a function, when averaged over the random point of exit of a particle in Brownian motion, solves the Laplace equation in the region.

Project 1: Participants, working in this group, will implement the random walk on spheres to find points of first encounter for a particle undergoing Brownian motion in regions of various shapes (half-plane, triangle, discs). Participants will experiment with how conditions at different points of the edge of the region may have vastly different effect on the value of the harmonic function (e.g. temperature) at a particular point.

Participants will determine optimal radii of circles as they iterate the steps of the algorithm. Matlab will be used to implement the algorithm and to present data. Some background in statistics is appreciated but not necessary, as all background material will be introduced by the research leader. The participants will develop a deep understanding of MCMC, core concepts of probability, and numerical methods.

Project 2: After implementation of RWOS, the participants will investigate an alternative way to simulate Brownian motion by the random jumps on a uniform Cartesian grid. The participants will compare convergence, accuracy, and efficiency of the two methods, as well as study errors due to discretization. More specifically participants will study effect of the grid step on the approximated solution.

References:

1. Deaconu and Lejay, *A Random Walk on Rectangles Algorithm*, Method. Comput. Appl. Probab.. **8**:1, 135-151 (2006).

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3. M. E. Muller, *Some continuous Monte Carlo methods for the Dirichlet problem*, Ann. Math. Statist. **27** pp. 569-589 (1956).
4. K.K. Sabelfeld and N.A. Simonov, Random Walks on Boundary for solving PDEs, 1994, ISBN 90-6764-183-9.