Knot theory, combinatorics, and grid diagrams
by
Teena Gerhardt
Department of Mathematics
Michigan State University
East Lansing, MI 48824-1027, USA
tena@math.msu.edu

Knot theory is the mathematical study of the way that strings can get tangled up and knotted. While often regarded as a subfield of topology, the past 30 years have seen an explosion of interest in this area which is in large part due to the remarkable interaction it has with other disciplines. In addition to topology and geometry, knot theory has applications and interaction with number theory, physics, biology, operator algebras, combinatorics, statistical mechanics, and discrete mathematics.

This project will introduce students to knot theory through its interactions with combinatorics and discrete mathematics. It is extremely useful to be able to encode the complexity of a knot with a finite amount of combinatorial data, and there are many ways to go about doing this. Historically, the first way is through so-called knot diagrams, which are pictures representing a knot which is tangled up in 3-dimensional space. A famous theorem of Reidemeister from 1926 gives a set of moves for passing between any two knot diagrams for the same knot [4]. These Reidemeister moves have been a fundamental tool for studying knots since their introduction. More recently, however, a more combinatorial device has been introduced for the study of knots called a grid diagram. Roughly, a grid diagram is a diagram of a knot which has the property that it is a union of horizontal and vertical line segments with the rule that vertical line segments always pass over horizontal segments. Grid diagrams have appeared in various guises throughout the history of knot theory, but have gained a considerable amount of notoriety over the past twenty years, most notably in the work Cromwell [1], Dynnikov [2], and Manolescu, Ozsváth, and Sarkar [3]. One of the key features of grid diagrams is that they allow for the encoding of a knot with a minimal amount of discrete data. Indeed, the only important feature about a grid diagram is the relative horizontal and vertical positions of the endpoints of the line segments comprising it, and these can be encoded with a pair of elements in the symmetric group on $n$ letters (where $n$ is the number of line segments). Just as with the Reidemeister moves, there are three simple transformations which allow one to pass between any two grid diagrams.

Students will study combinatorial aspects of knot theory through grid diagrams and other discrete tools. Possible avenues for investigation will be understanding the minimal number of vertical and horizontal segments for various families of knots (the so-called “grid number” of knots), investigating the relationship of grid diagrams with stick number, exploring colorability properties of knots through grid diagrams, and understanding the relationship between the combinatorics of a grid diagram for a knot and the pair of permutations which specifies it.
REFERENCES


