Permutation Statistics and q-analogues of the Catalan numbers

Tim Dwyer

Abstract

We call $\pi \in S_k$, where S_k is the symmetric group, a pattern in $\sigma \in S_n$ if σ has a subsequence which is in the same relative order as π, we might also say that σ contains π. For instance the permutation 3124 contains 213 as a pattern, since the terms 324 appear in the same relative order, but does not contain the pattern 321. A natural consideration then would be, for any $\pi \in S_k$, what $\sigma \in S_n$ do, or do not, contain π. With this in mind we define

$$Av_n(\pi) = \{\sigma : \sigma \text{ does not contain } \pi\}.$$

An astonishing result in the field of patterns is that for all $\pi \in S_3$ we actually have $|Av_n(\pi)| = C_n$ where the C_n are the famous Catalan numbers.

A function $st : S_n \to \mathbb{N}$. is called a statistic or a permutation statistic and we consider generating functions of the form

$$f(\pi; q) = \sum_{\sigma \in Av_n(\pi)} q^{st(\sigma)}.$$

We will investigate the properties of the above generating function when $st(\sigma)$ is the inversion number of σ (the number of entries of σ that appear out of their natural order), as a q-analogue of the Catalan numbers and relate them to various other q-Catalan numbers which have appeared in the literature.