Cops and Robbers in Tunnels

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Abstract

Cops and Robbers is a perfect information, vertex-pursuit game played on a finite reflexive graph G. Two players, a set of cops C and a robber R, begin by occupying vertices in G. The players then alternate moves (C then R), either moving along edges or passing. The cops win if at least one cop can move to the same vertex as the robber in a finite number of moves (called *capture*). Otherwise the robber wins. The *copnumber* of a graph G, denoted $c(G)$, is the minimum amount of cops needed to guarantee that the robber is captured. A graph G is called *copwin* if $c(G) = 1$. We consider joining two finite graphs G and H with a single edge, called a *tunnel*, to form a graph T. We classify the copwin graphs G and H such that $c(T) = 2$. We then partially classify the k-copwin graphs for which $c(T) = \max\{c(G), c(H)\}$.

I Introduction

Let G be a finite reflexive graph. Two players, a set of cops C and a robber R, play a vertex-pursuit game over a countable sequence of discrete time-steps called *rounds* [3]. In round 0, first C then R occupy some vertices in G. The players then alternate moves (C then R). To move in a round, each player must move to a neighboring vertex or *pass* and thereby stay on his current vertex. The cops win if at least one cop can move to the same vertex as the robber in a finite number of rounds. This is called *capture*. Otherwise the robber wins.

The game of cops and robbers was first discovered by Quilliot [10] and then independently by Nowakowski and Winkler [9] per G. Gabor. Both only considered the case of a single cop. The introduction of multiple cops and the notion of copnumber was made by Aigner and Fromme [1]. The *copnumber*, denoted $c(G)$, is the minimum amount of cops needed to guarantee that the robber is captured. A graph G is called *copwin* if $c(G) = 1$.

We quickly introduce some notation. We write $v \sim w$ if v is adjacent to w. Further, we call the set of vertices adjacent to a vertex v its *neighborhood* and denote it $N(v)$. If we wish to include v in its own neighborhood, we write $N[v]$.

A complete characterization of copwin graphs has already been completed. To see how, consider the second to last move of the cop, where the cop is on vertex C and the robber on vertex R. The robber could pass, so C must be adjacent to R (i.e., $C \sim R$). Or he could move to a neighboring vertex, so C must be joined to every neighbor of R. That is, $N[R] \subseteq N[C]$. Such a vertex R is called a pitfall.

Definition 1. [9]. A vertex v is a **pitfall** if there is some vertex w such that $N[v] \subseteq N[w]$. We write $v \to w$ and say that w **dominates** v.

Figure 1 gives an example of a pitfall R with $R \to C$.

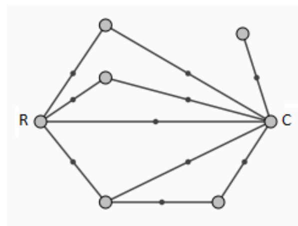

Figure 1: Pitfall R

Clearly pitfalls are useless for the robber. But every copwin graph must contain a pitfall.

Lemma 1. *[9]. If G is a copwin graph, then G contains at least one pitfall.*

Investigating the penultimate move of the cop, as in the previous paragraph, provides a proof of the lemma. We also need the following definition:

Definition 2. [3] An induced subgraph H of G is a **retract** of G if there is a homomorphism f from G to H so that $f(x) = x$ for $x \in V(H)$. The map f is called a **retraction**.

Berarducci and Intrigila [2] give the following result:

Theorem 2. *If H is a retract of G then $c(H) \leq c(G)$.*

For the purpose of this paper, we only consider retractions that *delete* (or *reduce*) a pitfall by mapping it to its dominating vertex. Thus, we only consider retracts that are compositions of deleting pitfalls. We call a graph *dismantlable* if some composition of deleting pitfalls yields K_1. We call K_1 the *trivial* retract.

Theorem 3. *[9] A graph is copwin if and only if it is dismantlable.*

We can order the vertices of a copwin graph such that the jth vertex is the jth pitfall deleted when dismantling the graph to K_1. We call this an *elimination ordering*. Section II explores elimination orderings and their realizations by spanning trees and Hasse diagrams.

The remaining sections deal with what we call tunnels. Let G and H be finite graphs and let $v \in V(G)$ and $w \in V(H)$. Form a graph T such that $V(T) = V(G) \cup V(H)$ and $E(T) = E(G) \cup E(H) \cup \{v, w\}$. We call the edge $\{v, w\}$ a *tunnel* and the resulting graph T a *tunneling* from (G, v) to (H, w). We insist that the direction of the tunnel is irrelevant. We work to classify the graphs for which $c(T) = \max\{c(G), c(H)\}$. It is not obvious that without the added cop the robber could not move across the tunnel from G to H indefinitely.

Section III covers tunneling between copwin graphs. In it, we classify vertices of copwin graphs to which tunneling would make $c(T)$=2. We then attempt to characterize the copwin graphs that have such vertices.

Section IV provides results for k-copwin tunnels and gives examples of k-copwin graphs for which $c(T) = \max\{c(G), c(H)\}$. Section V discusses variations of tunneling. Specifically, we explore tunneling within a copwin graph and building multiple tunnels between a series of graphs. Section VI motivates further work and concludes.

II Elimination Orderings and Hasse Diagrams

Every copwin graph has an elimination ordering. We now give a formal definition:

Definition 3. [3] An **elimination ordering** is a bijection $V(G) \to [n]$ that identifies a vertex by a positive integer in $[n]$ such that for each $i < n$, the vertex i is a pitfall in the subgraph induced by $\{i, i+1, \ldots, n\}$.

It is natural to extend this idea to a corresponding sequence in which the kth term is that which dominates the kth vertex of the elimination ordering. For a graph H, we write $H \upharpoonright S$ for the subgraph of H induced by the set of vertices S [3]. Assume that $[n]$ is an elimination ordering of G, and for $1 \leq i \leq n$ define

$$G_i = G \upharpoonright \{n, n-1, \ldots, i\}.$$

Note that $G_1 = G$ and G_n is just the vertex n. For each $1 \leq i \leq n-1$, let $f_i : G_i \to G_{i+1}$ be the retraction map from G_i to G_{i+1} mapping i onto a vertex that dominates it in G_i [3]. Necessarily, $f_i(i) > i$.

Definition 4. A **domination sequence** on a copwin graph G is a sequence of the form $f_i(i)$, $1 \leq i \leq n$, where $V(G) \to [n]$ is an elimination ordering and $f_i : G_i \to G_{i+1}$ is a choice of retraction.

We claim that given a domination sequence of a copwin graph G, there exists a subgraph H of G with that domination sequence. First we make the following definition:

Definition 5. [8] A **spanning tree** of an undirected graph G is a subgraph that includes all the vertices of G that is a tree.

And now for the result:

Theorem 4. *Given an elimination ordering and corresponding domination sequence of a copwin graph G, there exists a spanning tree H of G with that elimination ordering and domination sequence.*

Proof. We construct a graph H as follows: Begin with vertex n from the elimination ordering of G. Then consider vertex $n-1$ and attach it to $f_{n-1}(n-1) = n$. Continue to $n-2$, $n-3$, and so forth, adding i and attaching it to $f_i(i)$ for each vertex $i < n$. Since $f_i(i) > i$, we know that we can always attach a vertex to the graph at each step. First we show that there are no cycles. Assume there is a cycle formed by i, j, and k such that $i < j$ and $i < k$. So i is attached to two vertices with a larger index. By our construction algorithm, however, there is no vertex i that is attached to more than a single j for $j > i$. Thus, there can be no cycles and the constructed graph H is a tree. Next we show that H is a spanning tree. This follows because we proceed one-by-one through the elimination ordering, so all vertices will be reconstructed. □

This holds only insofar as we are given a copwin graph and choice of retractions. We can, however, realize *any* sequence of $n-1$ natural numbers in $[n]$ such that $a_i > i$ for $1 \leq i \leq n-1$ as a domination ordering.

Theorem 5. *Given $a_1, a_2, \ldots, a_{n-1} \in [n]$ such that $a_i > i$ for $1 \leq i \leq n-1$, there is a tree T with $V(T) = \{1, 2, \ldots, n\}$ such that $[n]$ is its elimination ordering with domination ordering $a_1, a_2, \ldots, a_{n-1}$.*

Proof. We construct T as follows: begin with vertex n. Then consider vertex $n-1$ and attach it to $a_{n-1} = n$. Continue to $n-2$, $n-3$, and so forth, adding i and attaching it to a_i for each $i < n$. Since $a_i > i$, we know that we can always attach a vertex to the graph at each step. First we show that there are no cycles. Assume there is a cycle formed by i, j, and k such that $i < j$ and $i < k$. So i is attached to two vertices with a larger index. By our construction algorithm, however, there is no vertex i that is attached to more than a single j for $j > i$. Thus, there can be no cycles and the constructed graph T is a tree. By reversing the construction, we find that $[n]$ is a valid elimination ordering with domination ordering $a_1, a_2, \ldots, a_{n-1}$. □

Here we only require that $a_i = f_i(i) > i$. So for each i, $f_i(i)$ can take any value from $i+1$ to n. That is, there are $n-i$ possible values for $f_i(i)$. For the general elimination ordering $[n]$, then, there are $(n-1)!$ possible domination orderings—all of which are realizable by a tree.

The set of elimination orderings can also be represented by another class of graphs: Hasse diagrams. To understand how, we first must recognize that an elimination ordering is a strict poset.

Definition 6. [11]. A **strict partial order** on a set S is a relation $\rho \subseteq S \times S$ that is irreflexive and transitive. The pair (S, ρ) is referred to as a **strict poset**.

The set of elimination orderings is a strict poset of the set $V(G)$ for some copwin graph G with the relation $<$ such that, for vertices $v, w \in V(G)$, $v < w$ if v occurs to the left of w in every elimination ordering of G. That is, v must *always* be eliminated before w.

Definition 7. [11] An element $a \in S$ is the **maximal element** of the poset $(S, <)$ if there is no element $x \in S$ such that $a < x$.

For example, the vertex n of the elimination ordering $V(G) \to [n]$ of a copwin graph G is a maximal element of the poset $(V(G), <)$. By the transitivity of a strict partial order, we also have the following:

Theorem 6. *[11] Every strict partial order is acyclic.*

We say that w **covers** v if $v < w$ and there does not exist $x \in V(G)$ such that $v < x < w$. We can now define a Hasse diagram:

Definition 8. [5] For the strict poset $P = (S, <)$, a **Hasse diagram** is an upward-directed acyclic graph H that has a vertex for each element in S and a directed edge $\{v, w\}$ for each pair (v, w) such that w covers v in P.

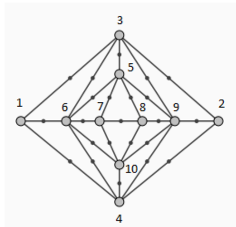

Figure 2: Copwin Graph G

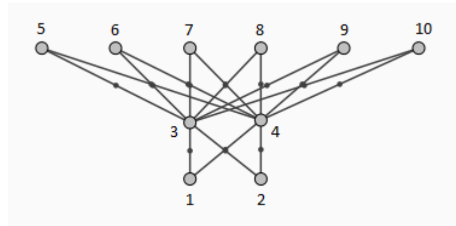

Figure 3: Hasse Diagram H

The set of elimination orderings of the copwin graph G in Figure 2 is represented as a Hasse diagram H in Figure 3. Notice that any vertex to which G can be dismantled is at the top of H. This follows from H being upward-directed.

Moreover, any vertex at the top of a Hasse diagram is a maximal element of the corresponding poset. So the vertices atop H in Figure 3 are the maximal elements of the poset $(V(G), <)$ for graph G in Figure 2. It is necessary for the work of Section III to give the following definition:

Definition 9. A vertex u is **fixed** if u is *not* a maximal element of the poset $(V(G), <)$, where $<$ is the relation such that, for vertices $v, w \in V(G)$, $v < w$ if v occurs to the left of w in every elimination ordering of G.

We will show in the next section that a copwin graph G can be dismantled to a vertex $v \in V(G)$ if and only if v is not fixed. We first, however, must turn to our concept of tunnels.

III Copwin Tunnels

Recall that a *tunnel* is an edge $\{v, w\}$ formed by joining graphs G and H at some vertices $v \in V(G)$ and $w \in V(H)$. The resulting graph is called a *tunneling*, denoted T, and we say that T is formed by tunneling from (G, v) to (H, w). More formally, T is a graph such that $V(T) = V(G) \cup V(H)$ and $E(T) = E(G) \cup E(H) \cup \{v, u\}$. We call G and H *components* of the tunneling and v and w the *endpoints* of the tunnel.

Lemma 7. *Form T by tunneling from (G, v) to (H, w). Then*

$$\max\{c(G), c(H)\} \leq c(T) \leq \max\{c(G), c(H)\} + 1.$$

Proof. Clearly $c(T) \geq \max\{c(G), c(H)\}$ because the robber can move exclusively in G or in H. We must have $c(T) \leq \max\{c(G), c(H)\} + 1$, for you could simply place the added cop on either endpoint of the tunnel, thereby containing the robber in either G or H. $\square$

For this section we assume that the components are both copwin and are therefore dismantlable. Consider the tunneling T in Figure 4, where the components are copwin triangles joined by a tunnel from 3 to 4. Clearly $\{1, 2, \ldots, 6\}$ is an elimination ordering of T, so T is copwin.

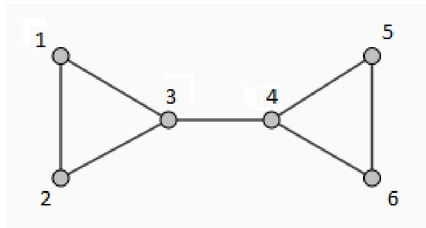

Figure 4

More generally, we notice that if a component graph G can be reduced to its respective endpoint of the tunnel v, then $\{V(G-v), v, V(H)\}$ will be an elimination ordering of T. That T is copwin immediately follows.

Can every copwin graph be reduced to any one of its vertices? We can quickly answer in the negative. We call a vertex $p \in V(G)$ a *unique pitfall* if p is the only pitfall in some nontrivial retract of G. In other words, at some point in an elimination ordering of G, the vertex p is the only possible choice to be reduced next. The top vertex of the graph in Figure 5 is a unique pitfall.

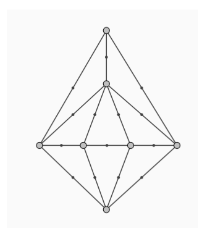

Figure 5

Here we can recall the definition of a fixed vertex and show that every unique pitfall is fixed. First we need the following lemma:

Lemma 8. *Let $p \in V(G)$ be the unique pitfall of a retract H of a copwin graph G. Then H is unique.*

Proof. Let H and H' be retracts in which p is the only pitfall and suppose $H \neq H'$. Then there is some vertex $x \in V(H)$ such that $x \notin V(H')$. Let $f : V(G) \to [n]$ be the elimination ordering where $f(x) = j$ is minimal. If $j > 1$, there are vertices $v_1, \ldots, v_{j-1}$ with $f(v_i) = i$ that are eliminated before x. Clearly $v_1, \ldots, v_{j-1} \notin V(H')$. We claim that $v_1, \ldots, v_{j-1} \notin V(H)$. Since p is a unique pitfall in H, $v_1 \notin V(H)$ because v_1 is a pitfall in G. If v_1 is deleted, then v_2 would be a pitfall, so $v_2 \notin H$. The same reasoning holds for all v_i for $1 \leq i \leq j-1$. So $v_1, \ldots, v_{j-1} \notin V(H)$. But this implies x is a pitfall in H, a contradiction. So $H = H'$. $\square$

Theorem 9. *Let G be a copwin graph and $p \in V(G)$ be a unique pitfall. Then p is fixed.*

Proof. Let H be the nontrivial retract of G in which p is the only pitfall. We know H is unique by Lemma 8. Since $|V(H)| > 1$ and p is the only pitfall, there exists some $x \in V(G)$ such that $p < x$. □

Even more is true. Every fixed vertex is also a unique pitfall of some copwin graph.

Theorem 10. *Let $v \in V(G)$ be fixed for some copwin graph G. Then v is a unique pitfall in some retract of G.*

Proof. Consider the retract H of G formed by deleting all vertices x such that $x < v$, then all vertices $y \neq v$ such that neither $y < v$ nor $v < y$. This is possible by definition of the relation $<$. The retract H is then the induced subgraph $G \restriction \{v, z_1, \ldots, z_k\}$ with $v < z_i$ for $1 \leq i \leq k$. Clearly v is a unique pitfall of H. □

So every unique pitfall is a fixed vertex and vice versa. We can now prove the claim that ended Section II:

Theorem 11. *Let G be copwin. Then G cannot be dismantled to $v \in V(G)$ if and only if v is fixed.*

Proof. Assume v is fixed. Then v is not a maximal element of the poset $(V(G), <)$. So there exists some $x \in V(G)$ such that $v < x$. It follows that G cannot be dismantled to v.

Now assume v is *not* fixed. Then v is never the only pitfall of any nontrivial retract of G. Let H be the retract of largest order in which v is a pitfall and let $|V(H)| = n$. Suppose $v \to d$ for some $d \in V(H)$. Since v is never the only pitfall, there exists some $w \in V(H)$ such that $v \neq w$ and w is a pitfall in H.

We show by cases that v is pitfall of $H - w$. **(1)** If $w \notin N(v)$, then eliminating w does not affect v, so $v \to d$ in $H - w$. **(2)** If $w \in N(v)$ but $w \neq d$, then $w \in N(d)$, so eliminating w does not change that $N[v] \subseteq N[d]$ since w is simply removed from both sets. Thus $v \to d$ in $H - w$. **(3)** If $w = d$, then $d \to z$ at stage for some $z \in V(H)$. It follows that $N[v] \subseteq N[d] \subseteq N[z]$. Thus, $v \to z$ in stage $H - w$. **(4)** Lastly, consider $d \to v$ in H. Then $N[v] = N[d]$, in which case either **(4a)** v and d are the only vertices left and are joined by a single edge or **(4b)** there is a $w \in N(v)$ such that w is a pitfall of H, returning us to the second case. For if there was not, then the elimination of v would leave the induced subgraph on $N[d] - v$ with more than one vertex and no pitfalls (d is no longer dominated by anything), a contradiction of G being copwin. This exhausts all cases. It follows that v is a pitfall in $H - w$.

So given a retract of G of order n in which v is a pitfall, we can delete a pitfall $w \neq v$ to form a retract of order $n - 1$ in which v is still a pitfall. We can continue this process until we obtain a retract of order $n = 2$ that must contain v. Clearly that retract can then be dismantled to v. So G can be dismantled to v. □

Corollary 11.1. *Every copwin graph G such that $|V(G)| > 1$ has at least two non-fixed vertices.*

Proof. Suppose $V(G) = n > 1$ and $[n]$ is an elimination ordering of G. Then $G \restriction \{n-1, n\}$ is a path of unit length. It follows that $N[n-1] = N[n]$, so $n - 1 \to n$ and $n \to n - 1$. So G can be dismantled to either $n - 1$ or n. By the contrapositive of Theorem 11, neither is fixed. □

We now move to our main result.

Theorem 12. *Let G and H be copwin graphs and let $v \in V(G)$ and $w \in V(H)$. Let T be the tunneling from (G, v) to (H, w). Then $c(T) = 2$ if and only if v and w are fixed.*

Proof. Suppose v and w are fixed. We first show that $c(T) \leq 2$. Place two cops on v. If the robber locates in G, keep one cop on v and let the other capture the robber in G since $c(G) = 1$. If the robber locates in H, move both cops to w. On the next turn, keep one cop on w and let the other capture the robber in H since $c(H) = 1$. Now we show that $c(T) > 1$. Consider the retracts G' and H' in which v and w are unique pitfalls, respectively. Form T' by tunneling from (G', v) to (H', w). So T' is a retract of T. Since there is no vertex $z \in V(T')$ such that $z \in N(v) \cap N(w)$, then neither v nor w is a pitfall in T'. But no other vertex is a pitfall either: v and w were unique pitfalls in G' and H', respectively. Thus, T' has no pitfalls and cannot be copwin by Theorem 3. It follows from Theorem 2 that T cannot be copwin.

For the forward direction, we prove the contrapositive. Suppose v is not fixed. Then by Theorem 11 there is an elimination ordering of G denoted $g : V(G) \to [n]$ with $g(v) = n$. Following this elimination ordering yields the induced subgraph of T on $V(H) \cup \{v\}$. Now $v \to u$, so v can in fact be eliminated in stage n. This leaves only H, which has an elimination ordering $h : V(H) \to [m]$ by assumption. Thus, the bijection that maps vertices of G to $[n]$ according to g and vertices of H to $[m] + n$ according to h is an elimination ordering of T. So T is copwin. $\square$

Figure 6 shows a tunneling between two fixed vertices. The reader can confirm that there are no pitfalls of this graph but that two cops can always capture the robber.

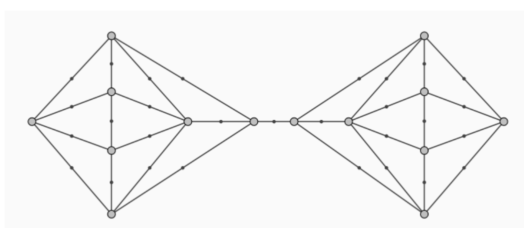

Figure 6

To this point, we can only identify the fixed vertices of a graph G by either drawing its Hasse diagram or attempting to dismantle G to each of its vertices. For the remainder of this section, we attempt to identify fixed vertices by the structures of the graphs of which they are members.

Recall the graph in Figure 5. This example is the smallest of an infinite class of graphs with a fixed vertex (see p.216-7 of [3] for more). All seem to share a specific structure in which the fixed vertex is part of an induced n-cycle, $n \geq 4$, and its dominating vertex is part of a nearly identical induced n-cycle. This leads to the following definition:

Definition 10. Let G be a finite connected graph and $p, v_1, \ldots, v_n \in V(G)$ with $n \geq 4$. An **induced pairwise cycle** $(p, v_1, \ldots, v_n)$ is an induced cycle $(v_1, \ldots, v_n)$ such that $p \sim$

v_1, v_2, v_n and $p \to v_1$ in some induced subgraph H of G. We call p the **peak** of the induced pairwise cycle.

Note that $(p, v_2, \ldots, v_n)$ is also an induced cycle–hence the name *pairwise.* We abbreviate induced pairwise cycle as IPC. Furthermore, we say an induced pairwise cycle satisfies the *empty neighborhood intersection (ENI) condition* if

$$\bigcap_{i=1}^{n} N[v_i] = \emptyset.$$

IPCs are defined for all finite connected graphs, not simply copwin graphs. Consider Figure 7, which contains a graph C whose only vertices are the induced pairwise 8-cycle.

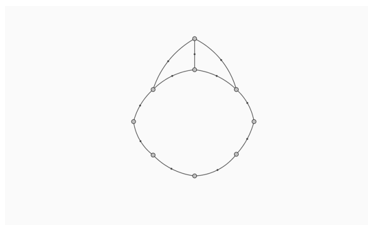

Figure 7

By the following lemma, C is not copwin:

Lemma 13. *[7] Suppose G is a graph with an induced cycle of length at least four, where at least one vertex of the cycle has degree two. Then G is not copwin.*

If we insist that a graph with an IPC be copwin, we can use Lemma 13 to prove the following theorem:

Theorem 14. *Let G be a plane copwin graph that satisfies the ENI condition for all induced pairwise cycles with peak $p \in V(G)$. Then p is a unique pitfall for some nontrivial retract H of G.*

Proof. We consider only retracts that are compositions of deleting pitfalls. Let G' be the retract in which all possible pitfalls are deleted such that no IPC with peak p is eliminated. Identify the outermost IPC and call these vertices $v_1, v_2, \ldots, v_{n-1}, v_n$ so that $p \to v_1$. Since the cycles are induced, we know the following hold: $p, v_1 \sim v_2$; $p, v_1 \sim v_n$; $p, v_1 \nsim v_i$ for $i \neq 1, 2, n$; and $v_i \nsim v_j$ for $j \neq i \pm 1$.

We first show that v_1, v_2, and v_n cannot be pitfalls in G'. Suppose there exists $z \in V(G)$ such that $v_1 \to z$. Then $\{p, v_1, v_2, v_n\} \subseteq N[v_1] \subseteq N[z]$. Consider the induced subgraph of G' on $\{p, z, v_1, v_2, \ldots, v_n\}$. Form the minor M by contracting v_i to v_{i-1} for $2 < i < n$. Then $M \cong K_5$. By Wagner's theorem [6], this contradicts the planarity of G. So there is no z such that $v_1 \to z$. A symmetric argument holds for v_2 and v_n, considering $N[v_2]$ and $N[v_n]$ and contracting v_i to v_{i-1} for $3 < i \leq n$ and $3 \leq i < n$, respectively.

By Lemma 13, for each i there must be some $w_i \neq v$ for any v such that $w_i \in N(v_i)$ and was not initially eliminated when forming G'. It is possible that $w_i = w_j$. Identify the outermost cycle $v_1, v_2, \ldots, v_{n-1}, v_n$ as above. Check one-by-one if $v_i \to w_i$ for $3 \le i \le n-1$. For each i such that $v_i \to w_i$, eliminate v_i and replace it with w_i. We claim the result is a k-cycle, $k \le n$. First, we show this is indeed a cycle. If $v_i \to w_i$, then $w_i \sim v_{i\pm1}$. If $v_{i-1} \to w_{i-1}$ and $v_{i+1} \to w_{i+1}$, then we must have $w_i \sim w_{i\pm1}$. Otherwise, at least one of v_{i-1} and v_{i+1} was not eliminated. But since $w_i \sim v_{i\pm1}$, we still have a k-cycle. Now we show $k \le n$. If $w_i = w_j$ for some $i \neq j$, then $k < n$. If $w_i \neq w_j$ for any $i \neq j$, then $k = n$. We cannot have $k > n$ since we replace v_i with only one vertex, w_i, that is necessarily adjacent to $v_{i\pm1}$. Now consider the new outermost k-cycle and rename these vertices $v_1, v_2, \ldots, v_{k-1}, v_k$. Then follow the process above, checking if $v_i \to w_i$ for the new corresponding $w_1, \ldots, w_k$.

Repeat this algorithm until either **(1)** no v_i is a pitfall or **(2)** no IPC with peak p remains. Call this retract H. If **(1)**, then H has p as its only pitfall and we are done. If **(2)**, consider H', the induced subgraph formed by the penultimate iteration of the algorithm. If $n = 4$, then for w such that $v_3 \to w$ we must have $w \sim v_2, v_3, v_4$. Since G is copwin, we must also have $w \sim v_1$ by Lemma 13. So w violates the ENI condition, a contradiction.

Otherwise, let $v_i \to w_i$ and $v_{i+1} \to w_{i+1}$. Then $w_i \sim v_{i-1}, v_i, v_{i+1}$ and $w_{i+1} \sim v_i, v_{i+1}, v_{i+2}$. It follows that either $w_i = w_{i+1}$ or $w_i \sim w_{i+1}$. But the latter contradicts planarity. To see this form the minor N as follows: contract $v_1, v_2, \ldots, v_{i-2}$ to v_{i-1} and $v_{i+3}, \ldots, v_n$ to v_{i+2}. Then delete all vertices other than $v_{i-1}, \ldots, v_{i+2}, w_i, w_{i+1}$. Lastly, delete the edges $\{v_i, w_i\}$ and $\{v_{i+1}, w_{i+1}\}$. Then $N \cong K_{3,3}$. By Wagner's theorem [6], this contradicts the planarity of G.

Thus $w_i = w_{i+1}$ for all $2 \le i < n$. So we have the 4-cycle $\{v_1, v_2, w, v_n\}$, where $w = w_i$, for all i. By Lemma 13, it must be that $v_1 \sim w$ for G to be copwin. Thus, w violates the ENI condition, a contradiction. This exhausts the possibilities for **(2)**, all of which lead to contradiction. So only **(1)** can occur. □

So if a copwin graph has an IPC with peak p, then p is a unique pitfall and is therefore fixed. We can create infinitely many examples of IPCs by embedding copwin graphs into such a cycle.

Theorem 15. *Let G be a plane copwin graph with n outer vertices and all pitfalls of G drawn outermost. If $\delta(G) \ge 3$, G can be embedded in a plane copwin graph G' with a unique pitfall.*

Proof. Let $\{v_1, \ldots, v_n\}$ be the outer vertices of G. We will embed G in a graph G' as follows: first, let $\{w_1, \ldots, w_n\}$ be the vertices of an induced n-cycle disjoint from G and attach v_i to w_i, $\forall i$. Next add p such that $p \sim w_1, w_2, w_n$ but $p \nsim w_j$ for $2 < j < n$. Thus, $p \to w_1$. If n is even: for $2 < j \le n/2$, add the edge $\{w_j, v_{j-1}\}$; for $n/2 \le j < n$, add the edge $\{w_j, v_{j+1}\}$. If n is odd: for $2 < j \le \lceil n/2 \rceil$, add the edge $\{w_j, v_{j-1}\}$; for $\lceil n/2 \rceil \le j < n$, add the edge $\{w_j, v_{j+1}\}$. Finally, add the edges $\{w_1, v_2\}$ and $\{w_1, v_n\}$. This is clearly a plane graph.

First we show that p is a unique pitfall in G'. By construction, $p \to w_1$. Also, w_1 is not a pitfall because only it is adjacent to both p and v_1. Furthermore, $w_i \not\to w_j$ and $v_i \not\to v_j$ for any i, j because the cycles are induced. Next we show that $v_i \not\to w_i$ for any i, $1 \le i \le n$. For any i, w_i is adjacent to at most three adjacent outer vertices in G (e.g., $w_1 \sim v_1, v_2, v_n$). Since $\delta(G) \ge 3$, we know v_i has some neighbor in G that is not $v_{i\pm1}$. Thus, $N[v_i] \not\subseteq N[w_i]$ for any i, $1 \le i \le n$.

It remains only to show that no w_j, $2 \leq j \leq n$, are pitfalls. Note that if w_j is a pitfall, it must be dominated by v_j and not any other v_i since no v_i is adjacent to more than two w_j, $i \neq j$. By construction, $w_j \sim w_{j\pm1}$. For $2 < j < \lfloor n/2 \rfloor$ and $\lceil n/2 \rceil < j < n$, we have $v_j \sim w_{j+1}$ or $v_j \sim w_{j-1}$ or neither, but not both. Thus $w_j \not\to v_j$ for $2 < j < \lfloor n/2 \rfloor$ and $\lceil n/2 \rceil < j < n$. If n is odd, the same holds for $w_{\lfloor n/2 \rfloor}$. For w_2 and w_n, it is the case that $v_j \sim w_{j\pm1}$, but we also have $w_2, w_n \sim p$ and $v_2, v_n \nsim p$. Thus, $w_2 \not\to v_2$ and $w_n \not\to v_n$. For $w_{\lceil n/2 \rceil}$ and w_1, the corresponding v_i are not adjacent to both w_{i-1} and w_{i+1}. Thus, no w is a pitfall, making p the unique pitfall.

Now we give an elimination ordering on $\{p, w_1, ..., w_n\}$ that will result in the original graph G. By what we showed above, such an elimination ordering must begin with p. With p eliminated, $N[w_2] \subseteq N[v_2]$ and $N[w_n] \subseteq [v_n]$, so w_2 and w_n can be eliminated next. For $2 \leq j < \lfloor n/2 \rfloor$ (and for $j = \lfloor n/2 \rfloor$ if n is odd), it is the case that $N[w_{j+1}] - \{w_j\} \subseteq N[v_{j+1}]$. So eliminating w_j makes $w_{j+1} \to v_{j+1}$. Likewise for $\lceil n/2 \rceil < j \leq n$, it is the case that $N[w_{j-1}] - \{w_j\} \subseteq N[v_{j-1}]$. So eliminating w_j makes $w_{j-1} \to v_{j-1}$. Eliminating all such vertices in these intervals leaves only w_1 and $w_{\lceil n/2 \rceil}$. But since all other w_j are eliminated, we have $w_1 \to v_1$ and $w_{\lceil n/2 \rceil} \to v_{\lceil n/2 \rceil}$. Thus, a valid elimination ordering is given by the following: $\{p, (w_2, w_n), (w_3, w_{n-1}), \ldots, w_1, (w_{\lfloor n/2 \rfloor}, w_{\lceil n/2 \rceil})\}$. This reduces G' to G. Since G was copwin, it follows that G' is copwin. □

Figure 8 gives an example of an embedding with $n = 6$. The embedded graph G is drawn with no inner vertices for the sake of generality.

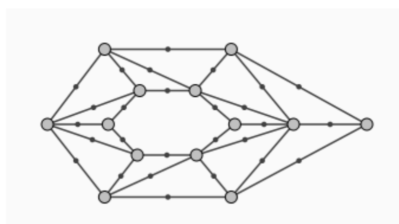

Figure 8

We have yet to find a graph with a fixed vertex that does not have an IPC. Thus, we wonder if any such graphs exist. We leave it to the reader to explore this.

IV k-Copwin Tunnels

We lack a characterization of k-copwin graphs for $k > 1$. Thus, we cannot give a complete characterization of the vertices to which one can or cannot tunnel. We can, however, identify graphs for which you can tunnel to *any* vertex and satisfy the lower bound of Lemma 7. Following the lead of Clarke et al. [4], we call a graph *neighborhood critical* if the removal of a vertex and its neighbors from the graph lowers its copnumber by 1. More formally:

Definition 11. A graph G with $c(G) = k$ is **neighborhood critical** if for $H = G - N[v]$ for any $v \in V(G)$, $c(H) = k - 1$.

Clearly no copwin graph is neighborhood critical. Cycles of length four or more, however, are neighborhood critical, since the corresponding H is a path. The class of neighborhood critical graphs also includes the Petersen and dodecahedral graphs. Figures 9 and 10 show that the Petersen graph is neighborhood critical.

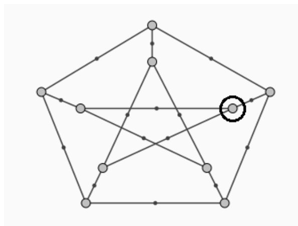

Figure 9: Petersen Graph

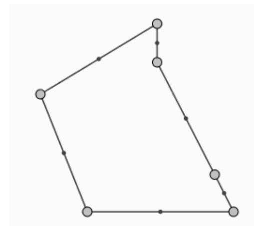

Figure 10: Deleted Neighborhood

We can now move to our results.

Theorem 16. *Let G_1 and G_2 be neighborhood critical graphs and tunnel from (G_1, v) to (G_2, w) to form T. Then $c(T) = \max\{c(G_1), c(G_2)\}$.*

Proof. Let $c(G_1) = m$ and $c(G_2) = n$ such that $n \leq m$. Since G_1 is a retract of T, we have $c(T) \geq m$ by Theorem 2. So we must show $c(T) \leq m$. Place m cops on v to start. If the robber begins in G_1, leave one cop on v and let the other $m - 1$ cops pursue him. Since one cop remains on v, the robber will never enter $N[v]$, lest he be caught on the next turn. So the game is played entirely on $H_1 = G_1 - N[v]$. By assumption, $c(H_1) = m - 1$, so the robber will be captured. If the robber begins in G_2, move all cops to w. On the next turn, leave one cop on w and let the other $m - 1$ cops pursue the robber. The robber will never enter $N[w]$, so the game is played entirely on $H_2 = G_2 - N[w]$. By assumption, $c(H_2) = n - 1 \leq m - 1$, so the robber will be captured. □

Corollary 16.1. *Let G be neighborhood critical with $c(G) = m$. Let K be a graph with $c(K) = n$ such that $n < m$. If we tunnel from (G, v) to (K, w) to form T, then $c(T) = \max\{c(G), c(K)\} = m$.*

Proof. Since G is a retract of T, we have $c(T) \geq m$ by Theorem 2. So we must show $c(T) \leq m$. Place m cops on v to start. If the robber begins in G, leave one cop on v and let the other $m - 1$ cops pursue him. Since one cop remains on v, the robber will never enter $N[v]$, lest he be caught on the next turn. So the game is played entirely on $H = G - N[v]$. By assumption, $c(H) = m - 1$, so the robber will be captured. If the robber begins in K, move all cops to w. On the next turn, leave one cop on w and let the other $m - 1$ cops pursue the robber. By assumption $c(K) = n < m$. Equivalently, $c(K) \leq m - 1$. So the robber will be captured. □

Our classification of k-copwin graphs to which we can tunnel is incomplete. We can, for example, tunnel between finite square grids (which have copnumber 2) to form T, and T will be 2-copwin. Such grids, however, are not neighborhood critical. We leave it open to further classify k-copwin graphs to which tunneling satisfies the lower bound of Lemma 7.

V Variations of Tunneling

We now consider variations of tunneling. We call a tunnel an *intratunnel* if it connects two previously unconnected vertices of the same graph. Thus, an intratunnel simply adds an edge to a graph.

Theorem 17. *Let p_1 and p_2 be pitfalls of a copwin graph G such that $N[p_1] \cap N[p_2] = \emptyset$. If we tunnel from p_1 to p_2 to form a graph T such that $V(T) = V(G)$ and $E(T) = E(G) \cup \{p_1, p_2\}$, then $c(T) = 2$.*

Proof. First we show that $c(T) \leq 2$. Place a cop on p_1 in T. Then the robber will never enter p_1 or p_2. So consider instead T' such that $V(T') = V(T)$ and $E(T') = E(T) - \{p_1, p_2\}$. Then $T' = G$, which is copwin. So $c(T) \leq 2$.

Next we show that $c(T) > 1$. Since $N[p_1] \cap N[p_2] = \emptyset$ in G, there is no vertex d that satisfies $d \sim p_1$ and $d \sim p_2$. Thus, there is no d such that $p_1 \to d$ or $p_2 \to d$ in any retract of T. So p_1 and p_2 can never be pitfalls and T can never be dismantled. $\square$

So tunneling within a copwin graph raises the copnumber by one if the neighborhoods of the endpoints are disjoint. This is true whether the endpoints of the tunnel are fixed or not. Figure 11 shows an intratunnel between two fixed vertices v and u, while Figure 12 shows one between two non-fixed vertices v and u.

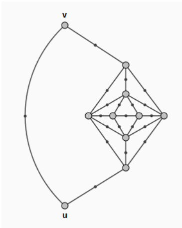

Figure 11: Fixed Vertices

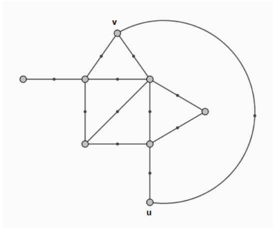

Figure 12: Non-fixed Vertices

An equivalent result cannot be obtained for k-copwin graphs, $k > 1$. Consider the 4-cycle in Figure 13. This is 2-copwin, but adding an intratunnel from v to u lowers the copnumber to 1. As for the graph in Figure 14, the reader can verify that its copnumber is also 2. But adding an edge from v to u yields the Petersen graph, which has copnumber 3.

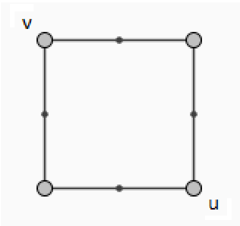

Figure 13

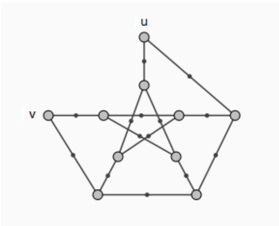

Figure 14

We can also consider tunneling between more than two graphs.

Definition 12. A **mine** M is a connected graph created by a series of tunnels from (G_i, v_i) to (G_j, v_j), $i \neq j$, for finite disjoint graphs $G_1, \ldots, G_k$ and vertices $v_i \in V(G_i)$. We call the induced subgraph $S = M \restriction \{v_1, \ldots, v_k\}$ the **skeleton** of the mine.

Note that each tunnel to G_j has v_j as its endpoint. Also note that the skeleton S is necessarily connected since the mine M is connected and the G_i are disjoint. Figure 15 shows a mine that connects four triangles $G_1, \ldots, G_4$ with the skeleton $S = C_4$.

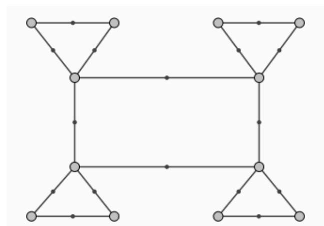

Figure 15

Theorem 18. *If G_i is copwin, $1 \leq i \leq k$, and $c(S) = n$, then $n \leq c(M) \leq n + 1$.*

Proof. We know $c(M) \geq n$ because the robber can simply stay in S. Thus, it remains only to show that $c(M) \leq n + 1$. Consider $n + 1$ cops placed in S. If the robber is in G_j, we say his *shadow* is at v_j. If he is in S at v_i, his shadow is also at v_i. Since the robber's shadow moves only in S, n cops can capture it. Insist that cops $c_1, \ldots, c_n$ play on S and assume they capture the shadow at v_j. Then the robber must remain in G_j. Send c_{n+1} into G_j. Since $c(G_j) = 1$, she will capture the robber in G_j. Thus, $c(M) \leq n + 1$. □

Corollary 18.1. *If G_i is copwin, $1 \leq i \leq k$, and $c(S) = n > 1$, then $c(M) = n$.*

Proof. It will suffice to show that $c(M) \leq n$. Consider n cops placed in S. Since $c(S) = n$, the robber's shadow will be caught in S. Assume cop c_i captures the shadow at v_i. If the robber is not on v_i, let cop c_j, $i \neq j$, enter G_i on the next possible turn. By assumption, c_j can capture the robber in G_i. □

For $n = 1$ either of the bounds on $c(M)$ can be realized. Both figures below have $n = 1$. The graph in Figure 16, however, is still dismantlable and thus has $c(M) = 1$; that in Figure 17 is not dismantlable and thus has $c(M) = 2$.

Corollary 18.1 holds even more generally:

Corollary 18.2. *Let $c(S) = n$. If $c(G_i) < n$, $1 \leq i \leq k$, then $c(M) = n$.*

Proof. It will suffice to show that $c(M) \leq n$. Consider n cops placed in S. Since $c(S) = n$, the robber's shadow will be caught in S. Assume cop c_i captures the shadow at v_i. If the robber is not on v_i, let all other cops enter G_i on the following turns. By assumption, $n - 1$ cops can capture the robber in G_i. □

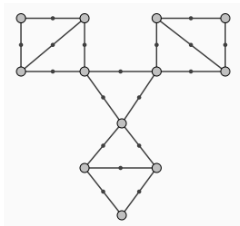

Figure 16: $c(M) = 1$

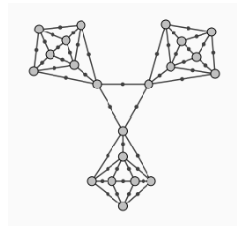

Figure 17: $c(M) = 2$

We have only considered tunneling to the same vertex v_i in each G_i when constructing a mine. Tunneling to multiple vertices of a G_i could allow the robber back into S once his shadow is captured at one v_i. So the upper bound on $c(M)$ in Theorem 14 will not hold. We leave it open whether such an upper bound can be found for the case of tunneling to multiple vertices of a G_i.

VI Further Work

Others have worked to characterize k-copwin graphs by decomposing graphs into subgraphs whose copnumbers are known (see Hill [7] and Clarke et al. [4]). We chose to work in reverse: by constructing a tunnel we create a larger graph from subgraphs with known copnumbers. We wonder if a complete classification of k-copwin graphs can be discovered by either of these methods.

We have given only a partial classification of the k-copwin graphs G and H for which $c(T) = \max\{c(G), c(H)\}$. So a complete classification is next. For copwin graphs, we wonder if every graph with a fixed vertex has an IPC. Lastly, we leave open further variations of tunneling.

VII Acknowledgements

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