Cops and Robbers on Infinite Graphs

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Abstract

Consider a connected graph G. The game of Cops and Robbers is played on G with two players C and R. C first places her cop(s) on the graph and then R places his robber on the graph. The two players alternate turns (C and then R) by either moving any number of their pieces to an adjacent vertex or staying still. C wins if a cop occupies the same vertex as R and R wins if he can always avoid capture. The weak cop number of a graph refers to the fewest number of cops it takes to guarantee that the robber never visits any vertex infinitely many times. We explore the weak cop number on products of infinite graphs as an extension of the work of Tošić, Maamoun, and Meyniel and discuss the extension of other results to the infinite setting.

1 Introduction

The game of Cops and Robbers on graphs has been explored by numerous mathematicians throughout recent decades. The game was introduced by Quilliot [12] in 1978 and independently by Nowakowski and Winkler [10] in 1983. Traditionally, it is played on a finite, connected, undirected graph G with two players C and R. C places some number of pawns, called cops, on whatever vertices of G she chooses, and R places a single pawn, the robber, on a vertex of his choice. The players alternate turns (C and then R) by moving any number of their pieces to adjacent vertices or staying still. Should one of the cops occupy the same vertex as the robber, C wins the game, and should the robber evade the cops indefinitely, R wins the game. Both players have perfect information and know where all pawns are on the graph. An excellent introduction to the game, together with the compiled results of numerous mathematicians, can be found in the book by Bonato and Nowakowski [2].

The *cop number* of a graph, denoted $c(G)$, is the fewest number of cops required to guarantee that C will win the game. This paper generalizes known results about the cop numbers of finite graphs to the infinite setting.

In exploring the game on infinite graphs, however, the notion of cop number alone is often less useful than in the finite case, as the robber frequently cannot be captured by a finite number of cops. Consider the infinite ray, P_∞. If the vertices of P_∞ are labeled $\{0, 1, 2, \ldots\}$, it is clear that, after the cops are placed on the graph, the robber can be placed on a vertex whose label is greater than that of any cop's vertex, and he will never be captured. See Fig.

1. (To avoid ambiguity, we use the notation $P_{\pm\infty}$ to denote a path which is infinite in both directions.)

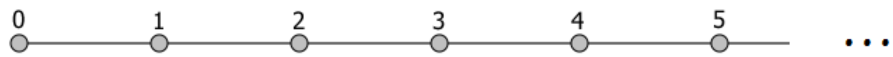

Figure 1: The infinite simple path, P_∞.

This effect is present in nearly all of the infinite graphs with which we are concerned, as shown in König's Lemma [3]:

Lemma 1 (Kőnig). *Every infinite locally finite connected graph contains an isometrically embedded copy of the infinite path P_∞.*

Locally finite refers to the property that every vertex is of finite degree. We focus our attention on locally finite graphs for simplicity and because the game is more natural on these graphs. Yet, because every infinite locally finite graph contains an isometric P_∞, no infinite locally finite graph has finite cop number, and we therefore introduce the following notion:

Definition 2 (Lehner). The **weak cop number** of a graph G, denoted $wc(G)$, is the minimum number of cops required to keep the robber from visiting any vertex infinitely many times.

This definition is due to Lehner [7], who improved on the original notion of weak cop number given by Chastand et. al. [4]. Lehner noted that the condition in Definition 2 is equivalent to stipulating that the robber must eventually leave every finite set of vertices. Another equivalent condition is the following:

Definition 3. The **weak cop number** of a graph G is the minimum number of cops required to guarantee that, for a chosen reference vertex v_0, the cops can force the robber to move along a sequence of vertices $(R_n)_{n=1}^{\infty}$ such that

$$\liminf_{n\to\infty} d_G(v_0, R_n) = \infty.$$

Note that by $d_G(a, b)$ we refer to the length of the shortest path in G between vertices a and b.

Proposition 4. *For locally finite graphs, definitions 2 and 3 are equivalent.*

Proof. We have already seen that the condition in Definition 2 equates to forcing the robber to eventually leave every finite set of vertices. If the cops force the robber into a sequence of moves $(R_n)_{n=1}^{\infty}$ such that he eventually leaves every finite set of vertices, then no matter the choice of v_0 we will have $\liminf_{n\to\infty} d_G(v_0, R_n) = \infty$ as the robber leaves the sets $\{v \in V(G) \mid d_G(v_0, v) \leq 1\}$, $\{v \in V(G) \mid d_G(v_0, v) \leq 2\}$, etc.

Conversely, if there is a finite set of vertices that the robber never leaves, and v_0 is chosen to be inside that set, then $\liminf_{n\to\infty} d_G(v_0, R_n)$ will be finite, as there will be a subsequence of $(R_n)_{n=1}^{\infty}$ for which $d_G(v_0, R_n)$ is finite. $\square$

We will use the condition found in Definition 3 in the following proofs and casually refer to the forcing of the robber along such a sequence as "chasing the robber infinitely far away from the reference vertex". It must be understood, however, that the definition requires not only that the robber moves infinitely far away from the reference vertex but that the robber eventually leaves every finite set of vertices and never returns.

Clearly, if G is a finite graph, $wc(G) = c(G)$. This fact allows us to generalize known results to the infinite setting - for instance, Corollary 8, which states that $wc(\square_{i=1}^n G_i) \leq \sum_{i=1}^n wc(G_i)$, applies not only to a set of infinite graphs G_i but also to set of finite graphs or a mix of infinite and finite graphs.

2 Cartesian products

The theorems in the following sections pertain to products of graphs. We will introduce their definitions and notations before we proceed.

Definition 5. The **Cartesian product** of two graphs G and H, notated $G \square H$, is the graph with vertex set $V(G) \times V(H)$ and edges drawn between two vertices (v_1, v_2) and (w_1, w_2) whenever

(a) v_1 is adjacent to w_1 in G and $v_2 = w_2$, or

(b) v_2 is adjacent to w_2 in H and $v_1 = w_1$.

We use the notation $\square_{i=1}^n G_i$ to describe $G_1 \square G_2 \square \cdots \square G_n$. See Figure 2 for an example of a Cartesian product.

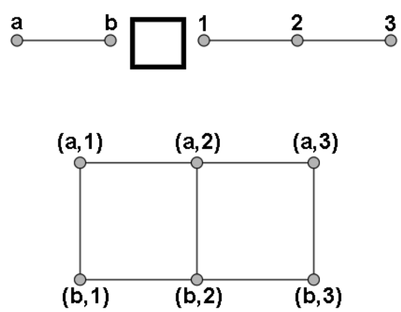

Figure 2: The Cartesian product of two graphs.

We use the following lemma from Hammack et. al. [5]:

Lemma 6 (Hammack, Imrich, and Klavžar). *For graphs G and H,*

$$d_{G \square H}((v_1, v_2), (w_1, w_2)) = d_G(v_1, w_1) + d_H(v_2, w_2).$$

We will now introduce terms that will be present throughout the rest of the paper, most of which are taken from Neufeld and Nowakowski [9]. We say that the product $G\Box H$ contains *copies of* G, defined to be induced subgraphs of $G\Box H$ with vertex set $\{(g,h_0) \mid g \in V(G)\}$ for some fixed $h_0 \in V(H)$. $G\Box H$ likewise contains copies of H.

By a *projection* of $G\Box H$ onto G_0, where G_0 is a copy of G within $G\Box H$, we refer to a map $f : G\Box H \to G_0$ which maps $(g,h) \mapsto (g,h_0)$. We say a cop has *captured the projection* of the robber in G_0 if, when the robber occupies $r \in G\Box H$, the cop occupies $f(r)$.

Finally, once a cop has captured the projection of the robber in G_0, we say that the cop *shadows* the robber if she moves such that, after the end of each of her turns, she has still captured the projection of the robber.

The following theorem is an extension to infinite graphs of a result of Tošić [13].

Theorem 7. *Given two connected locally finite graphs G and H, $wc(G\Box H) \leq wc(G) + wc(H)$.*

Proof. Let $wc(G) = m$ and $wc(H) = n$. We will show that $m+n$ cops are sufficient to either capture the robber on $G\Box H$ or chase him infinitely far away from a chosen reference vertex. First, let all $m+n$ cops be placed on some vertex $(g_0,h_0) \in V(G\Box H)$. They will seek to capture the projection of the robber onto the copy of G that they currently occupy, which we call G_0. Since $wc(G) \leq m+n$, these cops can either capture the robber's projection or chase the projection infinitely far away from (g_0,h_0).

However, if the robber's projection is moved infinitely far from (g_0,h_0), then the robber's position on $G\Box H$ will also move infinitely far from (g_0,h_0). This can be seen in the following inequality with Lemma 6 in mind, where (g_r,h_r) denotes the position of the robber on $G\Box H$:

$$d_{G\Box H}\left((g_r,h_r),(g_0,h_0)\right) = d_G(g_r,g_0) + d_H(h_r,h_0) \geq d_G(g_r,g_0) = d_{G\Box H}\left((g_r,h_0),(g_0,h_0)\right)$$

Therefore, we may assume that, rather than chasing the robber's projection in G_0 infinitely far from (g_0,h_0), the cops succeed in capturing his projection. Once the projection has been captured by a cop C_0, that cop can now shadow the robber. The other $m+n-1$ cops will continue to play on the same copy of G.

Once again, these cops will be sufficient to capture the robber's projection onto G_0 (assuming again that the robber is not chased infinitely far away), and hence another cop, C_1, will also be able to shadow the robber. In fact, this process can repeat until there are n cops who all shadow the robber on G_0. From then onward, whenever the robber makes a move along an edge in G, the n shadowing cops can continue to shadow, but when the robber moves along an edge in H, the n cops can play the game on whatever copy of H the robber occupies. As there are n cops playing on this copy of H, they will either capture the robber or force him infinitely far away from (g_0,h_0) should he decide to move along edges of H infinitely often. Hence, in order for the robber not to be captured or pushed infinitely far away, he must move along edges of H only finitely many times. After a finite amount of time, he will move exclusively on one copy of G.

Unfortunately for the robber, the other m cops (who were not among the shadowing cops) can move onto the same copy of G that the robber occupies while he is busy avoiding the first n cops by moving along edges of G. Once these m cops are all on the same copy of G as the robber, the game is practically over, as m cops are sufficient to either capture the robber or chase him infinitely far away from a reference vertex on any copy of G. $\Box$

The following corollary follows immediately from Theorem 7:

Corollary 8. *For a set of n connected locally finite graphs $\{G_i\}$, $wc(\square_{i=1}^n G_i) \leq \sum_{i=1}^n wc(G_i)$.*

We now explore in Theorem 13 a case in which $wc(G) + wc(H)$ is not an optimal bound for $wc(G \square H)$.

Definition 9. A **level order traversal** is a labeling of a tree such that one arbitrary vertex is designated to be the root and receives the label 0, and the rest of the vertices are labeled according to a breadth-first search of the tree. Specifically, the neighbors of vertex 0 are given labels $1, 2, \ldots, n$, while the vertices at distance two from vertex 0 are given labels $n+1, n+2, \ldots, n+m$, and so on.

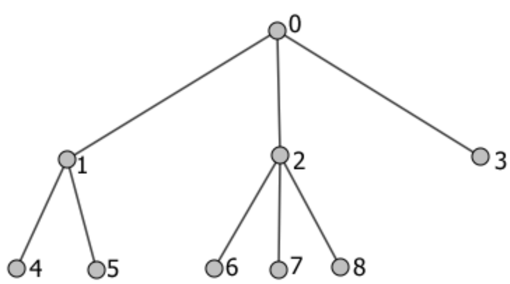

Figure 3: A tree labeled according to a level order traversal.

The following results, Lemma 10 and Theorems 11, 12, and 13 are extensions to infinite graphs of results of Maamoun and Meyniel [8]. The proofs of these results mirror arguments given by Maamoun and Meyniel, but there are a few modifications that are important and so we include them here.

Lemma 10. *For two locally finite trees T_1 and T_2, $wc(T_1 \square T_2) = 1$ provided that the robber is not allowed to stay on the same vertex indefinitely.*

Proof. Label the vertices of T_1 and T_2 according to a level order traversal of each tree. To capture the robber or chase him away from a reference vertex indefinitely, the cop can employ the following strategy: begin at the vertex $(0, 0)$. When $d_{T_1}(C, R) + d_{T_2}(C, R)$ is even, do nothing, and when $d_{T_1}(C, R) + d_{T_2}(C, R)$ is odd, move to decrease $\max(d_{T_1}(C, R), d_{T_2}(C, R))$. Clearly, no matter how the robber moves, the cop can guarantee that $d_{T_1 \square T_2}(C, R)$ has not increased from one of her turns to the next.

To see that this strategy is successful, let the position of C be given by (c_1, c_2) and let the position of R be denoted (r_1, r_2). From the onset, $c_1 < r_1$ and $c_2 < r_2$, and if R ever decreases r_1 or r_2 (by moving towards the root of some copy of T_1 or T_2), he will be decreasing the distance between himself and the cop. Since $d_{T_1 \square T_2}(C, R)$ never increases, R can only ever decrease r_1 or r_2 a finite number of times before he must increase r_1 or r_2 on every turn thereafter. This will cause the distance between R and $(0, 0)$, a convenient reference point, to increase monotonically without bound. The condition that R cannot stay on the same vertex indefinitely ensures that he will be forced to move further and further away from $(0, 0)$ eventually. $\square$

Theorem 11. *For a set of locally finite trees* $\{T_1, T_2, \ldots, T_n\}$*,* $wc(\square_{i=1}^n T_i) \leq \lceil \frac{n+1}{2} \rceil$*.*

Proof. We will proceed by induction on n. The first base case, when $n = 1$, is immediate given that $\square_{i=1}^1 T_i = T_1$ and the weak cop number of any tree is one. The second base case, when $n = 2$, follows from Theorem 7. For the induction hypothesis, assume that, for some value of $n \geq 3$, $wc(\square_{i=1}^n T_i) \leq \left\lceil \frac{n+1}{2} \right\rceil$ for any set of trees $\{T_i\}$. We will show that $wc(\square_{i=1}^{n+2} T_i) \leq \left\lceil \frac{(n+2)+1}{2} \right\rceil$.

Let $G = \square_{i=1}^{n+2} T_i$ for some set of trees $\{T_i\}$. To begin, let all $\left\lceil \frac{n+3}{2} \right\rceil$ cops begin on a vertex $v = (v_1, v_2, \ldots, v_{n+1}, v_{n+2}) \in V(G)$. Denote by $\hat{G}$ the induced subgraph with vertex set $\{(v_1, v_2, t_3, \ldots, t_{n+1}, t_{n+2}) \mid t_i \in V(T_i)\}$ — the copy of $\square_{i=3}^{n+2} T_i$ in G that the cops currently occupy. The cops will begin by attempting to capture the projection of the robber onto $\hat{G}$. By the induction hypothesis, either the cops can succeed in capturing this projection or the projection will necessarily move infinitely far from the reference point v. However, as explained in the proof of Theorem 7, if the robber's projection moves infinitely far away from v, so too will the robber himself. We may therefore assume that the robber's projection onto $\hat{G}$ is captured by a cop C_0.

From this point onward, C_0 will shadow the robber in $\hat{G}$. When the robber changes his first two coordinates, C_0 will follow the winning strategy on the copy of $T_1 \square T_2$ she occupies as described in Lemma 10. Should R change his first two coordinates infinitely many times, then, he will either be chased infinitely far away from v or be captured. He must therefore change his first two coordinates only finitely many times, at which point he will be restricted to a specific copy of $\square_{i=3}^{n+2} T_i$, and by the induction hypothesis, the remaining $\left\lceil \frac{n+1}{2} \right\rceil$ cops will be able to either capture him or force him infinitely far from v. The presence of these other cops also ensures that the robber cannot stay on the same vertex indefinitely, which satisfies the condition in Lemma 10.

$\square$

Theorem 12. *For a set of locally finite trees* $\{T_1, T_2, \ldots, T_n\}$*,* $wc(\square_{i=1}^n T_i) > \left\lceil \frac{n-1}{2} \right\rceil$*.*

Proof. We will describe a winning strategy for the robber against $\left\lceil \frac{n-1}{2} \right\rceil$ cops. Let the robber begin on some vertex that is distance at least 2 away from any cop. This is possible by Lemma 1. Now we prove, by induction on the number of turns t that have transpired, that at the end of the robber's turn he can always occupy a vertex of distance at least 2 away from any cop. (The base case, when $t = 1$, is trivial.)

Assume that, at the beginning of some turn t, $d(C_i, R) \geq 2$ for each cop C_i. If none of the cops move such that they are adjacent to the robber, then the robber need not move. Assume, therefore, that after the cops move there is at least one cop that is adjacent to the robber. Let $r = (r_1, r_2, \ldots, r_n)$ denote the position of the robber. Consider $C_{\leq 2}$, the set of all the cops at distance ≤ 2 from r. The position of each cop in $C_{\leq 2}$ will differ from the coordinates of r in at most 2 places. The fact that there are only $\left\lceil \frac{n-1}{2} \right\rceil$ cops playing on the graph, together with the fact that there is at least one cop whose position differs from r in only 1 place, ensures that there is some coordinate that r and the positions of all cops in $C_{\leq 2}$ have in common.

Now, for each coordinate r_i in the robber's vertex r, we will define a partner vertex $r_i' \in T_i$ by arbitrarily selecting one neighbor of r_i in T_i. (We will also define $(r_i')' = r_i$.)

Now, after the cops move, R can identify the coordinate that he has in common with every cop in $C_{\leq 2}$, say r_x, and move from $(r_1, r_2, \ldots, r_x, \ldots, r_n)$ to $(r_1, r_2, \ldots, r'_x, \ldots, r_n)$. Making this move will increase the distance to any cop in $C_{\leq 2}$ by one, ensuring that after R's move they will all still be of distance ≥ 2 from R. Furthermore, since before the robber's move we had $d(C_j, R) \geq 3$ $(\forall\ C_j \notin C_{\leq 2})$, we will guarantee that at the beginning of turn $t+1$, $d(C_i, R) \geq 2$ for all cops C_i.

Since R's moves always consist of switching one coordinate to that coordinate's partner vertex and back again, R will never be chased infinitely far away from his starting vertex. In fact, the furthest he may ever be from his starting vertex is at distance n. Thus, $\left\lceil \frac{n-1}{2} \right\rceil$ cops are not sufficient to capture him or force him infinitely far from his starting vertex. $\square$

Theorem 13. *For a set of locally finite trees* $\{T_1, T_2, \ldots, T_n\}$*,* $wc(\square_{i=1}^n T_i) = \left\lceil \frac{n+1}{2} \right\rceil$.

Proof. Theorems 11 and 12. $\square$

The weak cop numbers of some well-known infinite graphs follow immediate from these results. For example,

Corollary 14. *The n-dimensional regular infinite grid has weak cop number* $\left\lceil \frac{n+1}{2} \right\rceil$.

Proof. The n-dimensional regular infinite grid is isomorphic to $\square_{i=1}^n P_{\pm\infty}$, and $P_{\pm\infty}$ is a locally finite tree, so Theorem 13 applies. $\square$

3 Strong products

Definition 15. The **strong product** of two graphs G and H, notated $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H)$ and edges drawn between two vertices (v_1, v_2) and (w_1, w_2) whenever

(a) v_1 is adjacent to w_1 in G and $v_2 = w_2$, or

(b) v_2 is adjacent to w_2 in H and $v_1 = w_1$, or

(c) v_1 is adjacent to w_1 in G and v_2 is adjacent to w_2 in H.

Similar to the Cartesian product, we use $\boxtimes_{i=1}^n G_i$ to denote $G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_n$. See Figure 4 for an example of a strong product.

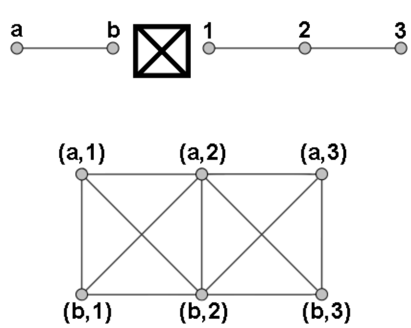

Figure 4: The strong product of two graphs.

We now generalize a result of Neufeld and Nowakowski [9]. We will use another lemma from Hammack et. al. [5]:

Lemma 16 (Hammack, Imrich, and Klavžar). *For graphs G and H,*

$$d_{G\boxtimes H}((v_1, v_2), (w_1, w_2)) = max(d_G(v_1, w_1), d_H(v_2, w_2)).$$

Theorem 17. *For two connected locally finite graphs G and H, $wc(G \boxtimes H) \leq wc(G) + wc(H) - 1$.*

Proof. Let $wc(G) = m$ and $wc(H) = n$. We will show that $m + n - 1$ cops are sufficient to weakly win on $G \boxtimes H$. First, let all of the cops be placed on some vertex $(g_0, h_0) \in V(G\Box H)$. As there are at least m of them, they can weakly win against the projection of the robber onto the copy of G that they currently occupy, denoted G_0. This means they can either capture the robber's projection or chase the projection infinitely far away from (g_0, h_0); however, chasing the projection infinitely far away will result in chasing the robber infinitely far away, as seen in the following inequality by Lemma 16 (where (g_r, h_r) represents the robber's position):

$$d_{G\boxtimes H}\left((g_r, h_r), (g_0, h_0)\right) = \max(d_G(g_r, g_0), d_H(h_r, h_0)) \geq d_G(g_r, g_0) = d_{G\boxtimes H}\left((g_r, h_0), (g_0, h_0)\right)$$

We therefore assume that the cops succeed in capturing his projection. Once the projection has been captured by a cop C_0, that cop can shadow the robber on G_0. Then, if there are still at least m cops who are not shadowing, another cop, C_1, will also be able to capture and subsequently shadow the robber. This process repeats until n cops have captured the robber's projection on G_0 (when $n - 1$ cops have captured the projection, there will still be m cops free, which is enough to have one more cop capture the projection).

Now, the n cops who have captured the robber's projection can continue to shadow the robber while simultaneously playing the game on a copy of H by changing their second coordinates according to whatever winning strategy exists on H. As there are n of them, they will be able to weakly win on a copy of H, which will cause them to weakly win on $G \boxtimes H$. □

4 Planar Graphs

The proofs of the major theorems so far have been, for the most part, relatively easily extended from finite graphs to infinite graphs. It may be unclear whether there are any results that do not extend easily, or at all, from the finite case to the infinite. The extension of the following result of Aigner and Fromme [1], however, is nontrivial. We will also see in section 5 a theorem, again by Aigner and Fromme, that is not true in the infinite case.

Theorem 18 (Aigner and Fromme). *We have $c(G) \leq 3$ for every planar graph G.*

The proof given by Aigner and Fromme considers 3 cops playing on G that use a strategy which, through the technique of guarding isometric paths in G, partitions G into the cop territory and the robber territory, and shows inductively that the cop territory can always be increased after a finite number of moves. In the case of finite graphs, the fact that the territory controlled by the cops can increase without bound guarantees that the entirety of G will eventually become cop territory and the robber will be captured.

In the case of infinite graphs, however, the fact that the cop territory becomes infinitely large over time does not guarantee the robber is caught, or even that the robber is chased infinitely far away from a reference vertex. Consider, for instance, the 2-dimensional regular infinite grid ($P_{\pm\infty}\Box P_{\pm\infty}$). The strategy given by Aigner and Fromme could allow the cops to control merely an infinite vertical strip centered around the cops' starting vertex. Should the robber be anywhere on the graph other than that vertical strip, he will never be caught nor forced to move. (Aigner and Fromme's proof will not be repeated here; we refer the reader to reference [1].) Extending Aigner and Fromme's result to infinite graphs, therefore, requires a more nuanced strategy for the cops. Our extension will nonetheless rely heavily on the methods and notation used in Aigner and Fromme's proof, and we recommend that the reader be familiar with their proof.

We first introduce definitions inspired by Komjáth and Pach [6] that will be useful in the following proof. Note that an **isometric subgraph** is a subgraph H of a graph G such that $d_H(u,v) = d_G(u,v)\,(\forall u,v \in V(H))$. A graph is To define the **boundary** of H , denoted ∂H, we fix a plane embedding of H and consider the complement of H in the plane. This complement consists of some finite number of simply connected regions, one of which is unbounded. ∂H refers to the induced subgraph of H on those vertices of H that are incident with the unbounded region in the plane.

Definition 19. An isometric subgraph H will be called **outer-geodesic** if, given any $a, b \in V(\partial H)$, $d_{\partial H}(a,b) \leq d_{(G-H)\cup\partial H}(a,b)$. Equivalently, H is outer-geodesic if ∂H is an isometric subgraph of $(G-H)\cup\partial H$.

Definition 20. A countable graph G is called **outer-geodesically constructible** if there exists a nested sequence of finite outer-geodesic subgraphs $G_1 \subseteq G_2 \subseteq G_3 \subseteq \cdots$ of G such that any $x \in V(G)$ is contained in some $V(G_i)$.

We now prove the following:

Theorem 21. *If G is countable, locally finite, planar, and outer-geodesically constructible, then $wc(G) \leq 3$.*

Proof. Assume G is countable, locally finite, planar, and outer-geodesically constructible. We will describe a winning strategy for 3 cops. Let $\{G_1, G_2, \ldots\}$ be finite outer-geodesic subgraphs of G with $G_1 \subseteq G_2 \subseteq \cdots$ and $\bigcup_{i=1}^{\infty} G_i = G$. The first step for the cops is to expand their territory such that all of G_1 is included within it. To accomplish this, the cops can play Aigner and Fromme's strategy with knowledge of the robber's whereabouts on the whole graph G but with the movement of the cops restricted to $V(G_1)$.

To verify that restricting the cops' movement does not prevent them from controlling all of G_1, we divide the situations present in Aigner and Fromme's proof into situation (a), the unlabeled first case of situation (b), and remaining cases of situation (b). In situation (a), R_i is the component of $G-u$ containing r, and for the purposes of controlling all of G_1, the cops can consider the robber territory to be $R_i \cap G_1$, which will make no difference in the cops' strategy because the cops will never be required to leave G_1 in order to use the strategy.

Likewise, in the first case of situation (b), the cops can consider the robber territory to be $R_i \cap G_1$ and their strategy will not be affected.

In the remaining cases of situation (b), the cops are concerned with guarding isometric paths that enclose the cop territory. As seen in Aigner and Fromme's Lemma 4, the process of guarding an isometric path from the robber involves moving onto the path and then along the path. According to that process, then, the restriction of the cops' movement to G_1 will not interfere with their ability to guard isometric paths from the robber. Recall that the cops still have full knowledge of the robber's whereabouts in G.

A complication that could conceivably arise in case (b) is that the paths P_1 and P_2 that the cops seek to control in G_1 are geodesics in $(R_i \cup P_1 \cup P_2) \cap G_1$ but not in $R_i \cup P_1 \cup P_2$. To prove that this problem never occurs, we rely on the planarity of G and that G_1 is an outer-geodesic subgraph in G.

Suppose, for later contradiction, that P_1 is a shortest u, v-path in $(R_i \cup P_1 \cup P_2) \cap G_1$ but that there is a shorter u, v-path $P_3 \subseteq R_i \cup P_1 \cup P_2$ containing some vertices from $V(G - G_1)$. By the planarity of G, P_3 must begin at vertex u and travel to a vertex $a \in V(\partial G_1)$, after which the path leaves G_1 before re-entering G_1 at a vertex $b \in V(\partial G_1)$ and continuing to v. (If P_3 leaves and re-enters G_1 multiple times, we let b denote the vertex the path reaches after re-entering G_1 for the last time). Let P_{3a} denote the section of P_3 going from u to a, let P_{3b} denote the section of P_3 going from b to v, and let $\hat{P}_3$ denote the rest of P_3. (See Figure 5.) Because G_1 is an outer-geodesic subgraph of G, there is an a, b-path P_4 lying entirely in $V(\partial G_1)$ such that $|P_4| \leq |\hat{P}_3|$. Hence, if we consider a new path given by the concatenation of P_{3a}, P_4, and P_{3b}, we will have found a u, v-path of length $\leq |P_3|$ lying entirely inside $(R_i \cup P_1 \cup P_2) \cap G_1$. This is a contradiction, however, given that P_1 was assumed to be minimal in $(R_i \cup P_1 \cup P_2) \cap G_1$ and P_3 was assumed to be shorter than P_1.

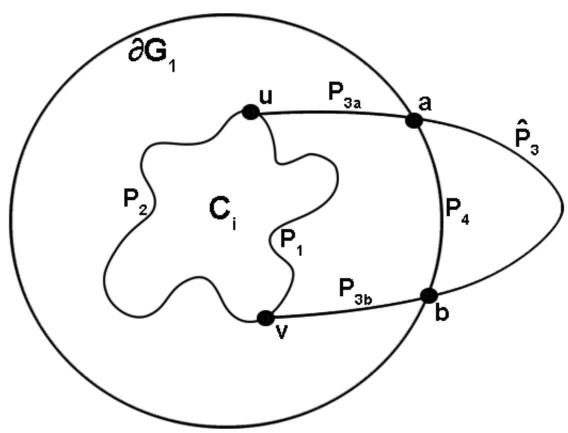

Figure 5: The fact that G_1 is outer-geodesic ensures that P_1 is a shortest path in G.

Once the cop territory includes all of G_1, the cops will then expand their territory to include all of G_2. This process will repeat indefinitely, eventually rendering every finite subset of G to be within the cop territory and securing a weak win for the cops. We need only verify that, once the cop territory includes all of, say, G_1, it can then be extended to all of G_2, and inductively extended to G_3, G_4, and so on.

We claim that, immediately after the cops expand their territory to include G_1, the conditions of the induction hypotheses found in Aigner and Fromme's proof are still met. This is easily verified on a case-by-case basis. As the conditions will be met each time a subgraph is completely assimilated into the cop territory, the cops will be able to control one subgraph after the other and eventually control the entire graph, meaning that there will never be a vertex that the robber can visit infinitely many times. □

We now complete the proof that $wc(G) \leq 3$ for every planar graph G. We will rely on the following result by Pach [6]:

Definition 22 (Pach). A countable graph G is called **isometrically constructible** if there exists an infinite sequence of finite isometric subgraphs of G: $G_1 \subseteq G_2 \subseteq G_3 \subseteq \ldots G$ such that any $x \in V(G)$ is contained in some $V(G_i)$.

Theorem 23 (Pach). *Every countable planar graph is isometrically constructible.*

In particular, Pach showed that every subgraph of a countable planar graph has an isometric supergraph. He did this by considering a subgraph, assuming it was not an isometric subgraph, adding a minimal improving path, and moving to the induced subgraph with those new vertices. If the new subgraph was still not an isometric subgraph, he repeated the process. Pach demonstrated, however, that this process eventually terminates, at which point the subgraph must be isometric. We will make use of this fact in the following proof.

Theorem 24. *Every countable planar graph is outer-geodesically constructible.*

Proof. To prove this statement, it is sufficient to prove that every isometric subgraph of a countable planar graph has an outer-geodesic supergraph. That is, if G_1 is an isometric subgraph of a countable planar graph G, then there is an outer-geodesic subgraph G_2 of G such that $G_1 \subseteq G_2$.

Consider a subgraph G_1 of G. Let the interior of G_1, denoted I, be given by $I = G_1 - \partial G_1$. Let the exterior of G_1, denoted E, be defined by $E = G - I$ (note that E includes vertices on the boundary of G_1). If G_1 is already an outer-geodesic subgraph of G, we're done; hence, we may assume ∂G_1 is not an isometric subgraph of E. However, if we ignore I and consider ∂G_1 as a subgraph of E, we know that ∂G_1 can be extended into an isometric subgraph of E by adding vertices and edges from E in the manner described by Pach [6]. We call this extended boundary (which is now an isometric subgraph of E) $(\partial G_1)^+$.

Let $G_2 = I \cup (\partial G_1)^+$. Clearly, $G_1 \subseteq G_2$. We claim that G_2 is an outer-geodesic subgraph of G. To demonstrate this, observe that $\partial G_2 \subset (\partial G_1)^+$ (NOTE: IF WE COULD PROVE ∂G_2 WAS AN ISOMETRIC SUBGRAPH OF $(\partial G_1)^+$, WE'D BE DONE).

□

5 Result that Doesn't Extend

Aigner and Fromme's minimum degree theorem... right? Assuming the weak cop number of the honeycomb is 2? (unfinished)

6 Future Work

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